# 臺灣二 OO 四年國際科學展覽會

科　　別：數學科

作品名稱：反正切函數，二階線性遞迴數列與疊在一起的
　　　　　方格紙

得獎獎項：數學科第一名
　　　　　美國第五十五屆國際科技展覽會團隊正選代表

學　　校：國立屏東高級中學
　　　　　高雄市私立道明高級中學

作　　者：郭子豪、李育霖

# 作 者 簡 介

我的名子叫做郭子豪，現就讀於屏東高中二年級。從小就對數學頗感興趣，但一直沒有深入研究，直到高中後才有機會較進一步的去探討數學的奧妙，也的確發現了數學的世界的無邊無際！

這次參加國際科展是我生平第一次，就由於是第一次，雖然在很多方面還是很生疏，但在更多的方面的確也下了不少苦心。終於看到自己的一份作品誕生，不論這份作品是否受到大家的青睞，對我而言都是一份無價的學習經驗！特別感謝張宮明老師的指導，以及同學和家人的支持。


我的名字叫李育霖，現就讀高雄市道明中學二年級，父母親都是數學教師，地地現就讀高雄中學。從小受父母親的薰陶，對數學情有獨鍾，特別是數學遊戲，也曾經做過幾次科展，都有不錯的佳績，國中時曾參加全國網路作文比賽榮獲第二名，也曾協助參與網界博覽會，榮獲全國第二名，國際銀牌獎，獲總統招見表揚。

這次是我第一次參與國際科學展覽，在參展過程中，使我獲益良多，也使我了解到，只要心存「好奇」，數學是個永無止盡的寶藏，等著你我去挖掘！

# 反正切函數，二階線性遞迴數列，與疊在一起的方格紙


中文摘要：

本文由三個結合 $\tan^{-1}$ 與費波那契數列的等式及其所搭配的無字證明圖形出發，做出和盧卡斯數列有關的圖形，並由數學歸納法找出並證明 $\tan^{-1}$ 與盧卡斯數列及一般二階線性遞迴數列的全新等式：

$$\arctan\frac{1}{L_{2i}}+\arctan\frac{1}{L_{2i+1}}=\arctan\frac{L_{2i+2}}{L_{4i+1}}$$

$$\arctan\frac{1}{L_{2i-1}}+\arctan\frac{1}{L_{2i}}=\arctan\frac{L_{2i+1}}{L_{4i-1}-2}$$

$$i\in N$$

$$\arctan\frac{1}{G_1}+\arctan\frac{1}{G_2}=\arctan\frac{G_3}{G_1G_2-1}$$

$$\arctan\frac{1}{G_i}+\arctan\frac{1}{G_{i+1}}=\arctan\frac{G_{i+2}}{\sum_{k=2}^{i}G_k^2+(G_1G_2-1)}$$

$$i\geq 2$$

而最後我們再討論等式 $\tan^{-1}\frac{1}{b}+\tan^{-1}\frac{1}{c}=\tan^{-1}\frac{1}{a}$ 之中的(a,b,c)之正整數解，我們利用電腦程式，求出其中之 100 組解，而後歸納其規律性如下：

| a | b | c |
|---|---|---|
| i | i+1 | $i^2$+i+1 |
| 2i+1 | 2i+3 | $2i^2$+4i+2 |
| $F_{2i}$ | $F_{2i+1}$ | $F_{2i+2}$ |

Abstract:

This paper startes with three equations of $\tan^{-1}$ and the Fibonacci sequence combined with the diagrams used to prove the three equations without words. According to the principle of mathematical induction, we continued to find out the similar equations of the Lucas numbers and the second-order liner recursive sequences as follows.

$$\arctan\frac{1}{L_{2i}}+\arctan\frac{1}{L_{2i+1}}=\arctan\frac{L_{2i+2}}{L_{4i+1}}$$

$$\arctan\frac{1}{L_{2i-1}}+\arctan\frac{1}{L_{2i}}=\arctan\frac{L_{2i+1}}{L_{4i-1}-2}$$

$$i\in N$$

$$\arctan\frac{1}{G_1}+\arctan\frac{1}{G_2}=\arctan\frac{G_3}{G_1G_2-1}$$

$$\arctan\frac{1}{G_i}+\arctan\frac{1}{G_{i+1}}=\arctan\frac{G_{i+2}}{\sum_{k=2}^{i}G_k^2+(G_1G_2-1)}$$

$$i\geq 2$$

In the end, we discussed the positive integer solutions of $\tan^{-1}\frac{1}{b}+\tan^{-1}\frac{1}{c}=\tan^{-1}\frac{1}{a}$ extended from the three equations of Fibonacci numbers.

We design a program to figure out the solutions and obtain the regulation as follows.

| a | b | c |
|---|---|---|
| i | i+1 | $i^2$+i+1 |
| 2i+1 | 2i+3 | $2i^2$+4i+2 |
| $F_{2i}$ | $F_{2i+1}$ | $F_{2i+2}$ |

## 壹、前言及研究動機：

我們在暑假讀到一篇有趣的數學文章[1]。在這篇僅僅兩頁的短文中，作者 Ko 介紹了三個結合了 $\tan^{-1}$ 與費波那契數列的奇妙等式：

$$\tan^{-1}\frac{1}{F_{2i}}=\tan^{-1}\frac{1}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+2}} \tag{1}$$

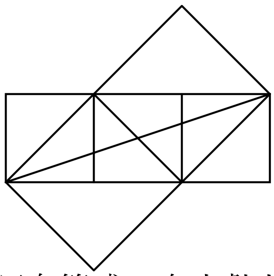
$$\tan^{-1}\frac{2}{F_{2i+2}}=\tan^{-1}\frac{1}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+4}} \tag{2}$$

$$\tan^{-1}\frac{1}{F_{2i}}=\tan^{-1}\frac{2}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+3}} \tag{3}$$

其中費波那契數列 $\{f_i\}_{i\geq 1}=\{1,1,2,3,5,8,13,21,\cdots\}$ 。

Ko 用畫圖方式給了這三個等式分別在最一開始各兩個初始值的“無字證明”(Proof without words)，因此共有六個無字證明。比如說，這個圖可以說明 $\tan^{-1}1=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$ ：

.



Ko 並且附註，據他所知只有等式(1)在文獻上有紀錄，(2)(3)似乎是新的等式。

這篇短文引起我們強烈的興趣。這三個等式居然能把 $\tan^{-1}$ 與費波那契數結合在一起，還能用無字證明來說明前幾個等式是對的，令我們覺得相當美麗。但是仔細一想，有許多問題這篇文章並沒有說明清楚，比如說，僅僅兩個初始值的畫圖證明並不能說明往後的值就會是對的，而且事實上等式(2)(3)也沒證明。因此開始了我們的研究。

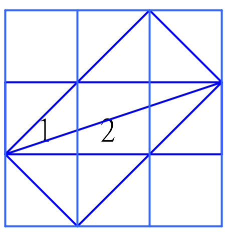
我們想瞭解文中的等式和無字證明是否真的正確。於是利用解析坐標幾何和代數不同的角度來看這個問題。

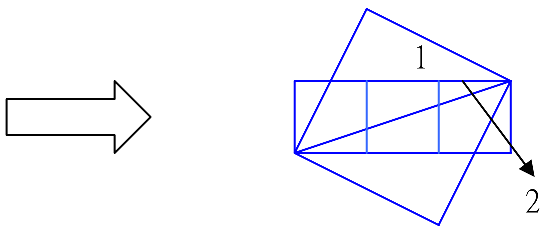
**首先我們發現：**

$\tan^{-1}$ 在方格紙上疊合的圖形，可以用不同方法把它疊合出來！

譬如說：

【1】 3×3 格




證明：

$\tan^{-1}\frac{1}{2}=\angle 1$

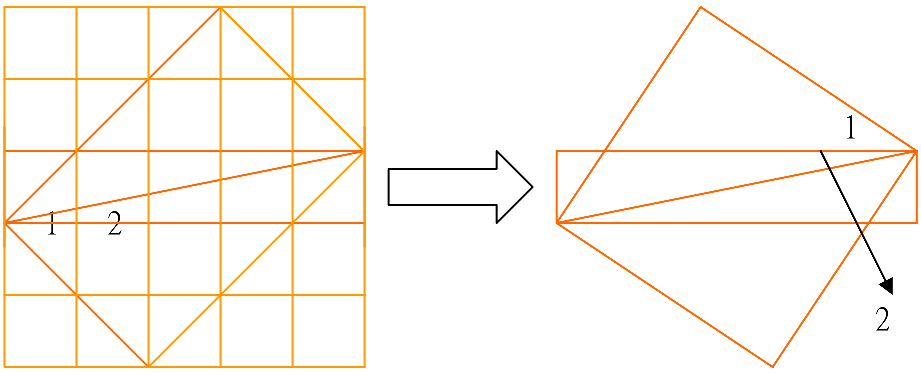
$$\tan^{-1}\frac{1}{3} = \angle 2$$

$$\angle 1 + \angle 2 = 45^0$$

$$\tan^{-1} 1 = 45^0$$

$$\therefore \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1} 1$$

【2】5×5 格




證明：

$$\tan^{-1}\frac{2}{3} = \angle 1$$

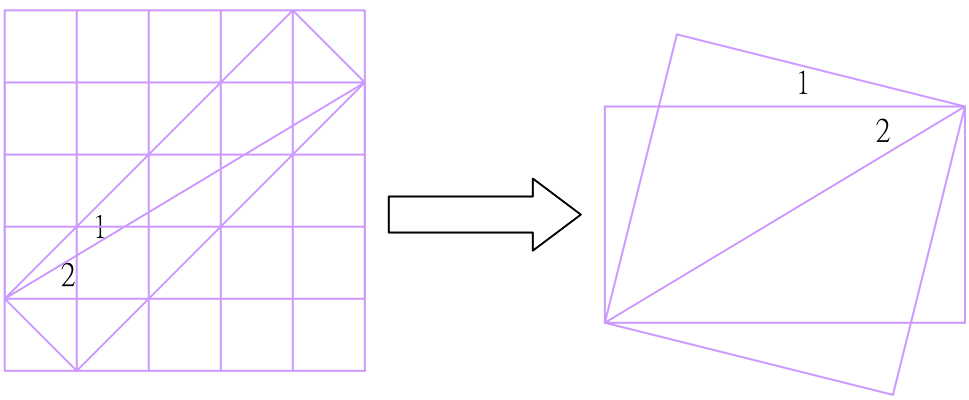
$$\tan^{-1}\frac{1}{5} = \angle 2$$

$$\angle 1 + \angle 2 = 45^0$$

$$\tan^{-1} 1 = 45^0$$

$$\therefore \tan^{-1} 1 = \tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{5}$$

【3】5×5 格



證明：

$\tan^{-1}\dfrac{1}{4}=\angle 1$

$\tan^{-1}\dfrac{3}{5}=\angle 2$

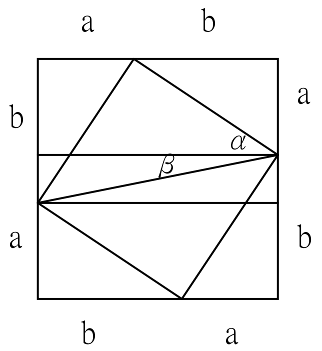
$\angle 1+\angle 2=45^0$

$\tan^{-1}1=45^0$

$\therefore \tan^{-1}1=\tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{3}{5}$

【4】$(a+b)\times(a+b)$格

證明




$\tan\alpha=\dfrac{a}{b}\Rightarrow\alpha=\tan^{-1}\dfrac{a}{b}$

$\tan\beta=\dfrac{b-a}{b+a}\Rightarrow\beta=\tan^{-1}=\dfrac{b-a}{b+a}$

$\tan(\alpha+\beta)=1\Rightarrow\alpha+\beta=\tan^{-1}1$

$\therefore \tan^{-1}\dfrac{a}{b}+\tan^{-1}\dfrac{b-a}{b+a}=\tan^{-1}1$

另一方面，如果這些等式((1)為已知，(2)(3)為新的等式)是對的，則他們應該不會是孤立的三個等式。因為如果費波那契數列有這三個等式，則相似的盧卡斯數列也應該有類似的等式。更進一步，既然費波那契數列與盧卡斯數列是最簡單的二次線性遞迴數列，所以一般的二次線性遞迴數列都應該有類似的等式才對。反過來說，如果二次線性遞迴數列都有類似的等式，而費波那契的情況可以用不同尺度的方格紙來疊合證明，則盧卡斯數列以及一般的二次線性遞迴數列也應該有方格紙的疊合證明。

我們的研究報告分成幾個小節。在第一節中，我們先確定等式(1)(2)(3)是對的，並且證明每一個等式都能有圖形證明。第二節中，我們將問題推廣到 Lucas 數列，得到一系列的關於 Lucas 數列與 $\tan^{-1}$ 結合的新等式。第三節中，我們更證明了任意的二階線性遞迴數列與 $\tan^{-1}$ 結合的嶄新等式。

貳、研究方法，過程及結果：

一、費波那契數列的 arctangent 等式，與方格紙的疊和：

首先我們證明等式(1)(2)(3)是對的，對於每個等式我們都給予數學歸納法及解析幾何之證明。

| |
|---|
| 定理 1 $\tan^{-1}\frac{1}{F_{2i}}=\tan^{-1}\frac{1}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+2}} \quad \forall i\in N$ |

定理 1 的代數證明：

證明 $\tan^{-1}\frac{1}{F_{2i}}=\tan^{-1}\frac{1}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+2}}$

1. i＝1 時 $\tan^{-1}\frac{1}{F_2}=\tan^{-1}\frac{1}{F_3}+\tan^{-1}\frac{1}{F_4}$ 成立

2. 設 i＝k 時成立 即 $\tan^{-1}\frac{1}{F_{2k}}=\tan^{-1}\frac{1}{F_{2k+1}}+\tan^{-1}\frac{1}{F_{2k+2}}$

同取 tan： $\frac{1}{F_{2k}}=\frac{\frac{1}{F_{2k+1}}+\frac{1}{F_{2k+2}}}{1-\frac{1}{F_{2k+1}F_{2k+2}}}=\frac{F_{2k+2}+F_{2k+1}}{F_{2k+2}F_{2k+1}-1}$

$\therefore 1=F_{2k+2}F_{2k+1}-F_{2k+2}F_{2k}-F_{2k+1}F_{2k}$

3. 當 i＝k+1 時

$$
\begin{aligned}
&F_{2k+4}F_{2k+3}-F_{2k+4}F_{2k+2}-F_{2k+3}F_{2k+2}\\
&=(F_{2k+3}+F_{2k+2})F_{2k+3}-(F_{2k+3}+F_{2k+2})(F_{2k+3}-F_{2k+1})-F_{2k+3}(F_{2k+1}+F_{2k})\\
&=F_{2k+3}{}^2+F_{2k+3}F_{2k+2}-F_{2k+3}{}^2-F_{2k+3}F_{2k+2}+F_{2k+3}F_{2k+1}+F_{2k+2}F_{2k+1}-F_{2k+3}F_{2k+1}-F_{2k+3}F_{2k}\\
&=F_{2k+2}F_{2k+1}-(F_{2k+2}+F_{2k+1})F_{2k}\\
&=F_{2k+2}F_{2k+1}-F_{2k+2}F_{2k}-F_{2k+1}F_{2k}=1
\end{aligned}
$$

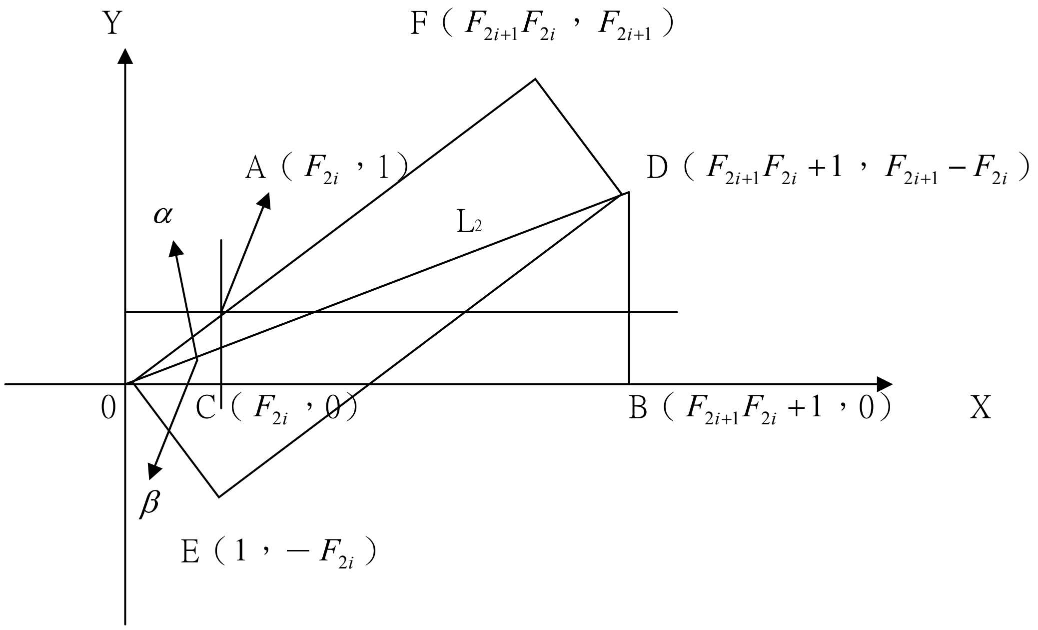
$$\therefore \frac{1}{F_{2k+2}}=\frac{\frac{1}{F_{2k+4}}+\frac{1}{F_{2k+3}}}{1-\frac{1}{F_{2k+4}F_{2k+3}}}$$

$$\Rightarrow \tan^{-1}\frac{1}{F_{2k+2}}=\tan^{-1}\frac{1}{F_{2k+3}}+\tan^{-1}\frac{1}{F_{2k+4}}$$

即 i＝k+1 時成立

故 $\tan^{-1}\frac{1}{F_{2i}}=\tan^{-1}\frac{1}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+2}}$

定理 1 的幾何證明




如上圖所示 OEDF 為長方形, 設 $\angle FOD=\alpha,\ \angle DOB=\beta$

直線 L2：y＝ $\dfrac{F_{2i+1}-F_{2i}}{F_{2i+1}F_{2i}+1}x$

tan（ $\alpha+\beta$ ）＝ $\dfrac{1}{F_{2i}}$　　$\therefore \alpha+\beta=\tan^{-1}\dfrac{1}{F_{2i}}$

在 $\Delta$ ODF 中　$\tan\alpha=\dfrac{\sqrt{F_{2i}{}^{2}+1}}{F_{2i+1}\sqrt{F_{2i}{}^{2}+1}}=\dfrac{1}{F_{2i+1}}$

$\therefore \alpha=\tan^{-1}\dfrac{1}{F_{2i+1}}$

在 $\Delta OBD$ 中　$\tan\beta=\dfrac{F_{2i+1}-F_{2i}}{F_{2i+1}F_{2i}+1}$　　$\therefore \beta=\tan^{-1}\dfrac{F_{2i+1}-F_{2i}}{F_{2i+1}F_{2i}+1}=\tan^{-1}\dfrac{1}{F_{2i+2}}$

$\Rightarrow \tan^{-1}\dfrac{1}{F_{2i}}=\tan^{-1}\dfrac{1}{F_{2i+1}}+\tan^{-1}\dfrac{1}{F_{2i+2}}$

定理 2

$$\tan^{-1}\frac{2}{F_{2i+2}}=\tan^{-1}\frac{1}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+4}}\quad \forall i\in N$$

定理 2 的代數證明：

1. i＝1 時 $\tan^{-1}\frac{2}{F_4}=\tan^{-1}\frac{1}{F_3}+\tan^{-1}\frac{1}{F_6}$

2. .設 i＝k 時成立　即 $\tan^{-1}\frac{2}{F_{2k+2}}=\tan^{-1}\frac{1}{F_{2k+1}}+\tan^{-1}\frac{1}{F_{2k+4}}$

同取 tan ：$\frac{2}{F_{2k+2}}=\frac{\frac{1}{F_{2k+1}}+\frac{1}{F_{2k+4}}}{1-\frac{1}{F_{2k+1}F_{2k+4}}}=\frac{F_{2k+1}+F_{2k+4}}{F_{2k+4}F_{2k+1}-1}$

$2F_{2k+4}F_{2k+1}-2=F_{2k+2}F_{2k+1}+F_{2k+2}F_{2k+4}$

$2F_{2k+4}F_{2k+1}-F_{2k+2}F_{2k+1}-F_{2k+2}F_{2k+4}=2$

3.當 i＝k+1 時

$2F_{2k+6}F_{2k+3}-F_{2k+4}F_{2k+3}-F_{2k+4}F_{2k+6}$

＝$2(F_{2k+5}+F_{2k+4})F_{2k+3}-(F_{2k+5}F_{2k+4})F_{2k+4}-F_{2k+4}F_{2k+3}$

＝$2F_{2k+3}(2F_{2k+4}+F_{2k+3})-(2F_{2k+4}+F_{2k+3})F_{2k+4}-F_{2k+4}F_{2k+3}$

＝$4F_{2k+3}F_{2k+4}+2F_{2k+3}{}^2-2F_{2k+4}{}^2-2F_{2k+4}F_{2k+3}$

＝$2F_{2k+4}F_{2k+2}+2F_{2k+4}F_{2k+1}+2F_{2k+3}{}^2-2F_{2k+3}{}^2-4F_{2k+3}F_{2k+2}-2F_{2k+2}{}^2$

＝$2F_{2k+4}F_{2k+1}+2F_{2k+4}F_{2k+2}-2F_{2k+3}F_{2k+2}-2F_{2k+3}F_{2k+2}-F_{2k+2}{}^2$

＝$-2F_{2k+3}F_{2k+2}+2F_{2k+4}F_{2k+1}$

＝$2F_{2k+4}F_{2k+1}-2(F_{2k+4}-F_{2k+2})F_{2k+2}$

＝$2F_{2k+4}F_{2k+1}-2F_{2k+4}F_{2k+2}+2F_{2k+2}{}^2$

＝$2F_{2K+4}F_{2K+1}-F_{2k+2}F_{2k+4}-F_{2k+2}F_{2k+4}+2F_{2k+2}{}^2$

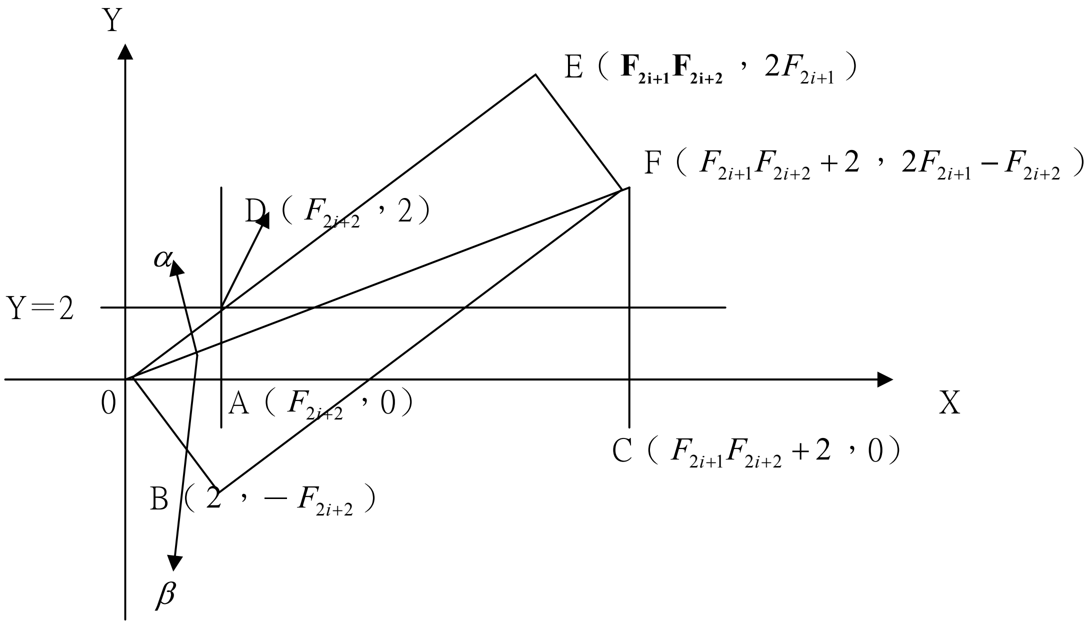
＝$2F_{2k+4}F_{2k+1}-F_{2k+2}F_{2k+4}-F_{2k+2}(2F_{2k+2}+F_{2k+1})+2F_{2k+2}{}^2$

＝$2F_{2k+4}F_{2k+1}-F_{2k+2}F_{2k+4}-2F_{2k+2}{}^2-F_{2k+2}F_{2k+1}+2F_{2k+2}{}^2$ ＝2　得證

故 $\tan^{-1}\frac{2}{F_{2i+2}}=\tan^{-1}\frac{1}{F_{2i+1}}+\tan^{-1}\frac{1}{F_{2i+4}}$

定理２的解析幾何證明

$$\tan^{-1}\frac{\mathbf{2}}{\mathbf{F_{2i+2}}}=\tan^{-1}\frac{\mathbf{1}}{\mathbf{F_{2i+1}}}+\tan^{-1}\frac{\mathbf{1}}{\mathbf{F_{2i+4}}}$$




令 $\overline{OD}$ ＝K

$$\tan(\alpha+\beta)=\frac{2}{F_{2i+2}}$$

$$\tan\alpha=\frac{K}{F_{2i+1}K}=\frac{1}{F_{2i+1}} \qquad \alpha=\tan^{-1}\frac{1}{F_{2i+1}}$$

$$\beta=\tan^{-1}\frac{2F_{2i+1}-F_{2i+2}}{F_{2i+1}F_{2i+2}+2}=\tan^{-1}\frac{1}{F_{2i+4}}$$ 故得證

定理３：

$$\tan^{-1}\frac{1}{F_{2i}}=\tan^{-1}\frac{2}{F_{2i+2}}+\tan^{-1}\frac{1}{F_{2i+3}} \quad \forall i\in N$$

定理３的代數證明：

1.i＝1 $\tan^{-1}\frac{1}{F_2}=\tan^{-1}\frac{2}{F_3}+\tan^{-1}\frac{1}{F_5}$ 成立

2. 設 i＝k 時成立，即 $\tan^{-1}\frac{1}{F_{2k}}=\tan^{-1}\frac{2}{F_{2k+2}}+\tan^{-1}\frac{1}{F_{2k+3}}$

同取 tan：$\frac{1}{F_{2k}}=\frac{\frac{2}{F_{2k+2}}+\frac{1}{F_{2k+3}}}{1-\frac{1}{F_{2k+2}F_{2k+3}}}=\frac{2F_{2k+3}+F_{2k+2}}{F_{2k+2}F_{2k+3}}=\frac{1}{F_{2k}}$

$$F_{2k+3}F_{2k+2}-2F_{2k+3}F_{2k}-F_{2k}F_{2k+2}=2$$

3. i＝k+1 時

$$F_{2k+5}F_{2k+4}-2F_{2k+5}F_{2k+2}-F_{2k+4}F_{2k+2}$$
$$=(F_{2k+4}+F_{2k+3})(F_{2k+3}+F_{2k+2})-2F_{2k+2}(F_{2k+4}+F_{2k+3})-F_{2k+2}(F_{2k+3}+F_{2k+2})$$
$$=F_{2k+4}F_{2k+3}+F_{2k+4}F_{2k+2}+F_{2k+3}{}^2+F_{2k+3}F_{2k+2}-2F_{2k+4}F_{2k+2}-2F_{2k+2}F_{2k+3}-F_{2k+3}F_{2k+2}-F_{2k+2}{}^2$$
$$=F_{2k+4}F_{2k+3}-F_{2k+4}F_{2k+2}-2F_{2k+2}F_{2k+3}+F_{2k+3}{}^2-F_{2k+2}{}^2$$
$$=F_{2k+3}(F_{2k+3}+F_{2k+2})-F_{2k+2}(F_{2k+3}+F_{2k+2})-2F_{2k+3}(F_{2k+1}+F_{2k})+F_{2k+3}{}^2-F_{2k+2}{}^2$$
$$=F_{2k+3}{}^2+F_{2k+3}F_{2k+2}-F_{2k+3}F_{2k+2}-F_{2k+2}{}^2-2F_{2k+3}F_{2k+1}-2F_{2k+3}F_{2k}+F_{2k+3}{}^2-F_{2k+2}{}^2-2F_{2k+3}F_{2k}$$
$$=2\left(F_{2k+3}{}^2-F_{2k+2}{}^2\right)-2F_{2k+3}F_{2k+1}-2F_{2k+3}F_{2k}$$
$$=2\left(2F_{2k+2}F_{2k+3}-2F_{2k+2}{}^2+F_{2k+3}F_{2k+1}-F_{2k+2}F_{2k+1}\right)-2(F_{2k+3}F_{2k+1})-2F_{2k+3}F_{2k}$$
$$=4F_{2k+2}F_{2k+3}-4_{2k+2}{}^2-2F_{2k+2}F_{2k+1}-2F_{2k+3}F_{2k}$$
$$=F_{2k+2}(4F_{2k+3}-4F_{2k+2}-2F_{2k+1})-2F_{2k+3}F_{2k}$$
$$=F_{2k+2}[4(F_{2k+2}+F_{2k+1})-4F_{2k+2}-2F_{2k+1}]-2F_{2k+3}F_{2k}$$
$$=F_{2k+2}(2F_{2k+1})-2F_{2k+3}F_{2k}$$
$$=F_{2k+2}(F_{2k+3}-F_{2k})-2F_{2k+3}F_{2k}$$
$$=F_{2k+3}F_{2k+1}-2F_{2k+3}F_{2k}-F_{2k}F_{2k+1}=2 \quad \text{得證}$$

故 $\tan^{-1}\dfrac{1}{F_{2i}}=\tan^{-1}\dfrac{2}{F_{2i+2}}+\tan^{-1}\dfrac{1}{F_{2i+3}}$

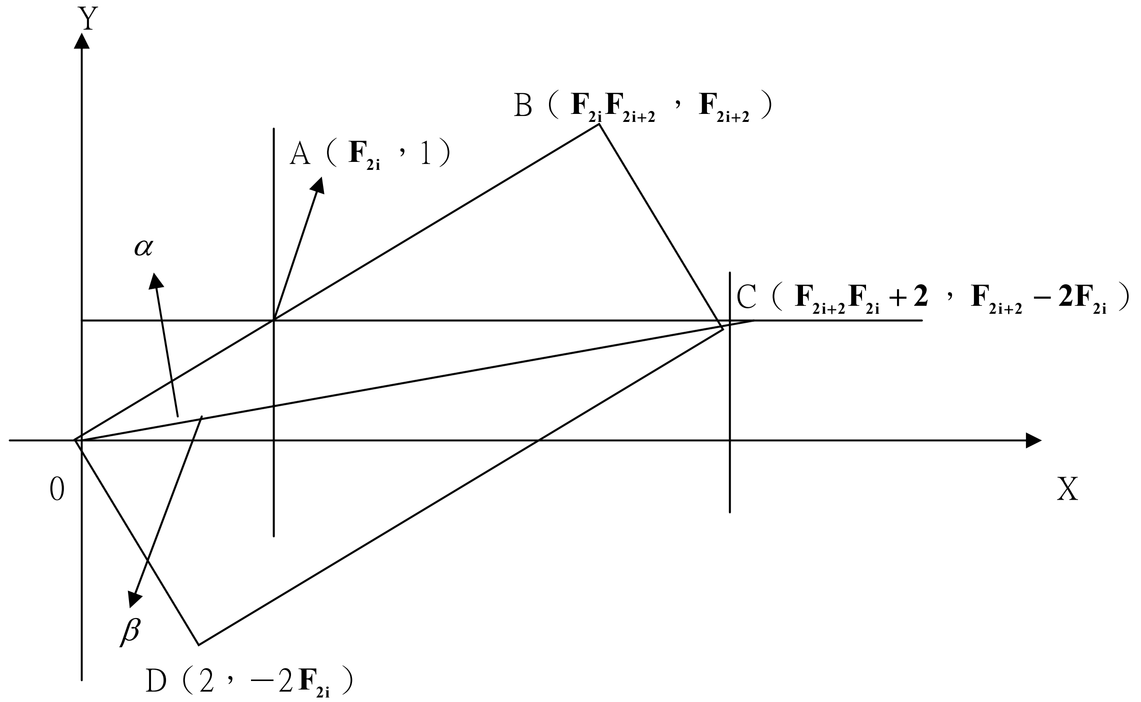
定理 3 的解析幾何證明：

令 $\overline{OA}$ ＝K

$$\tan(\alpha+\beta)=\frac{1}{F_{2i}}$$

$\tan\alpha=\dfrac{2K}{F_{2i+2}K}=\dfrac{2}{F_{2i+2}}$　　　$\therefore \alpha=\tan^{-1}\dfrac{2}{F_{2i+2}}$

$\boldsymbol{\beta=\tan^{-1}\dfrac{F_{2i+2}-2F_{2i}}{F_{2i+2}F_{2i}+2}=\tan^{-1}\dfrac{1}{F_{2i+3}}}$　　**故得證**




二、與盧卡斯數列有關的 arctangent 等式，與方格紙的疊和：

此節當中，我們尋找盧卡斯數列與 arctangent 的關係，並將其歸納成一般式，並運用三角函數和已知的盧卡斯數列性質做代數證明。

盧卡斯數列<$L_i$>定義：$\mathbf{L_1=1, L_2=3, L_{i+2}=L_i+L_{i+1}, i\geq 1, i\in N}$ ，其前十項如下表所示。

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_i$ | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 |

首先我們運用 tangent 的和角公式：$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

並令 $\tan\alpha=\dfrac{1}{3}, \tan\beta=\dfrac{1}{4}, \cdots$ ，來尋找關係。由此，我們有了二項嶄新的發現。

(1)

$$\mathbf{\arctan\frac{1}{L_2}+\arctan\frac{1}{L_3}=\arctan\frac{L_4}{L_{2+3}}}$$

$$\mathbf{\arctan\frac{1}{L_4}+\arctan\frac{1}{L_5}=\arctan\frac{L_6}{L_{4+5}}}$$

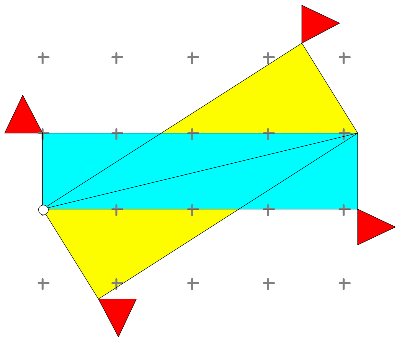
$$\mathbf{\arctan\frac{1}{L_6}+\arctan\frac{1}{L_7}=\arctan\frac{L_8}{L_{6+7}}}$$

$$\mathbf{\arctan\frac{1}{L_8}+\arctan\frac{1}{L_9}=\arctan\frac{L_{10}}{L_{8+9}}}$$

$$\mathbf{\arctan\frac{1}{L_{10}}+\arctan\frac{1}{L_{11}}=\arctan\frac{L_{12}}{L_{10+11}}}$$

………

$$\mathbf{Example: \arctan\frac{1}{3}+\arctan\frac{1}{4}=\arctan\frac{7}{11}}$$



(2)

$$\arctan\frac{1}{L_1}+\arctan\frac{1}{L_2}=\arctan\frac{L_3}{L_{1+2}-2}$$

$$\arctan\frac{1}{L_3}+\arctan\frac{1}{L_4}=\arctan\frac{L_5}{L_{3+4}-2}$$

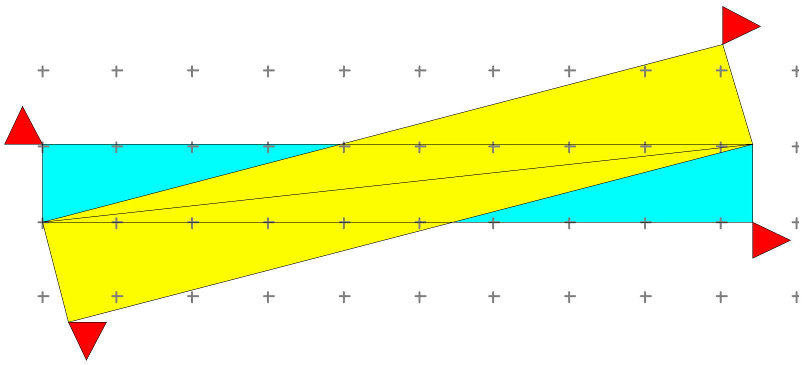
$$\arctan\frac{1}{L_5}+\arctan\frac{1}{L_6}=\arctan\frac{L_7}{L_{5+6}-2}$$

$$\arctan\frac{1}{L_7}+\arctan\frac{1}{L_8}=\arctan\frac{L_9}{L_{7+8}-2}$$

$$\arctan\frac{1}{L_9}+\arctan\frac{1}{L_{10}}=\arctan\frac{L_{11}}{L_{9+10}-2}$$

.........

$$\text{Example}:\arctan\frac{1}{4}+\arctan\frac{1}{7}=\arctan\frac{11}{27}$$



依據上述，我們大膽猜測盧卡斯數列與 arctangent 有關的等式。接下來我們用三角函數和盧卡斯數列的性質來證明。由已知的盧卡斯數列性質：$L_mL_n=L_{n+m}+(-1)^mL_{n-m}$ , n＞m [2] 可推得引理 4.1。

引理 4.1：
令<$L_k$>爲盧卡斯數列， $k \in N$ ，則

$$L_k L_{k+1} = L_{2k+1} + (-1)^k$$

定理 4：

$$\arctan\frac{1}{L_{2i}} + \arctan\frac{1}{L_{2i+1}} = \arctan\frac{L_{2i+2}}{L_{4i+1}} \qquad (1)$$

$$\arctan\frac{1}{L_{2i-1}} + \arctan\frac{1}{L_{2i}} = \arctan\frac{L_{2i+1}}{L_{4i-1}-2} \qquad (2)$$

$i \in N$

定理 4 的證明：首先證明第一式

$$\tan\left(\arctan\frac{1}{L_{2i}} + \arctan\frac{1}{L_{2i+1}}\right) = \tan\left(\arctan\frac{L_{2i+2}}{L_{4i+1}}\right)$$

$$\Leftrightarrow \frac{\tan\left(\arctan\frac{1}{L_{2i}}\right) + \tan\left(\arctan\frac{1}{L_{2i+1}}\right)}{1-\tan\left(\arctan\frac{1}{L_{2i}}\right)\tan\left(\arctan\frac{1}{L_{2i+1}}\right)} = \frac{L_{2i+2}}{L_{4i+1}}$$

$$\Leftrightarrow \frac{\frac{1}{L_{2i}} + \frac{1}{L_{2i+1}}}{1-\frac{1}{L_{2i}}\times\frac{1}{L_{2i+1}}} = \frac{L_{2i+2}}{L_{4i+1}}$$

$$\Leftrightarrow \frac{L_{2i}+L_{2i+1}}{L_{2i}L_{2i+1}-1} = \frac{L_{2i+2}}{L_{4i+1}}$$

$$\Leftrightarrow \frac{L_{2i+2}}{L_{2i}L_{2i+1}-1} = \frac{L_{2i+2}}{L_{4i+1}}$$

$$\Leftrightarrow L_{4i+1}+1 = L_{2i}L_{2i+1}$$

所以由引理**4.1**可得證。

再來證明第二式

$$\tan\left(\arctan\frac{1}{L_{2i-1}}+\arctan\frac{1}{L_{2i}}\right)=\tan\left(\arctan\frac{L_{2i+1}}{L_{4i-1}-2}\right)$$

$$\Leftrightarrow\frac{\tan\left(\arctan\frac{1}{L_{2i-1}}\right)+\tan\left(\arctan\frac{1}{L_{2i}}\right)}{1-\tan\left(\arctan\frac{1}{L_{2i-1}}\right)\tan\left(\arctan\frac{1}{L_{2i}}\right)}=\frac{L_{2i+1}}{L_{4i-1}-2}$$

$$\Leftrightarrow\frac{\frac{1}{L_{2i-1}}+\frac{1}{L_{2i}}}{1-\frac{1}{L_{2i-1}}\times\frac{1}{L_{2i}}}=\frac{L_{2i+1}}{L_{4i-1}-2}$$

$$\Leftrightarrow\frac{L_{2i}+L_{2i-1}}{L_{2i}L_{2i-1}-1}=\frac{L_{2i+1}}{L_{4i-1}-2}$$

$$\Leftrightarrow\frac{L_{2i+1}}{L_{2i}L_{2i-1}-1}=\frac{L_{2i+1}}{L_{4i-1}-2}$$

$$\Leftrightarrow L_{4i-1}-1=L_{2i}L_{2i-1}$$

所以由引理**4.1**可得證。

三、二階線性遞迴數列與 arctangent 有關的等式

此節中，我們定義一個新的遞迴數列 D(1,4)如下

數列<$D_i$>滿足 $\mathbf{D_1=1,D_2=4,D_{i+2}=D_{i+1}+D_i,i\geq 1,i\in N}$ 下表為其前 10 項。

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $D_i$ | 1 | 4 | 5 | 9 | 14 | 23 | 37 | 60 | 97 | 157 |

$$\arctan\frac{1}{D_1}+\arctan\frac{1}{D_2}=\arctan\frac{D_3}{D_2-D_1}=\arctan\frac{D_3}{3}$$

$$\arctan\frac{1}{D_2}+\arctan\frac{1}{D_3}=\arctan\frac{D_4}{D_2^2+3}$$

$$\arctan\frac{1}{D_3}+\arctan\frac{1}{D_4}=\arctan\frac{D_5}{D_3^2+D_2^2+3}$$

$$\arctan\frac{1}{D_4}+\arctan\frac{1}{D_5}=\arctan\frac{D_6}{D_4^2+D_3^2+D_2^2+3}$$

$$\arctan\frac{1}{D_5}+\arctan\frac{1}{D_6}=\arctan\frac{D_7}{D_5^2+D_4^2+D_3^2+D_2^2+3}$$

………

在 D(1,4)這個數列中所觀察出來的關係和在費波那契數列及盧卡斯數列中所觀察到的截然不同，但卻和我之前所猜測的“等式中會和數列第二項和第一項的差($D_2-D_1$)有關”，在數列<$D_i$>中所發現的關係，在費波那契數列及盧卡斯數列亦有相同的結果。這些我們所發現的嶄新等式如下：

1．更新費波那契數列的等式

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |

$$\arctan\frac{1}{F_1}+\arctan\frac{1}{F_2}=\arctan\frac{F_3}{F_2-F_1}=\arctan\frac{F_3}{0}$$

$$\arctan\frac{1}{F_2}+\arctan\frac{1}{F_3}=\arctan\frac{F_4}{F_2^2+0}$$

$$\arctan\frac{1}{F_3}+\arctan\frac{1}{F_4}=\arctan\frac{F_5}{F_3^2+F_2^2}$$

$$\arctan\frac{1}{F_4}+\arctan\frac{1}{F_5}=\arctan\frac{F_6}{F_4^2+F_3^2+F_2^2}$$

$$\arctan\frac{1}{F_5}+\arctan\frac{1}{F_6}=\arctan\frac{F_7}{F_5^2+F_4^2+F_3^2+F_2^2}$$

………

2．更新盧卡斯數列的等式

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_n$ | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 |

$$\arctan\frac{1}{L_1}+\arctan\frac{1}{L_2}=\arctan\frac{L_3}{L_2-L_1}=\arctan\frac{L_3}{2}$$

$$\arctan\frac{1}{L_2}+\arctan\frac{1}{L_3}=\arctan\frac{L_4}{L_2^2+2}$$

$$\arctan\frac{1}{L_3}+\arctan\frac{1}{L_4}=\arctan\frac{L_5}{L_3^2+L_2^2+2}$$

$$\arctan\frac{1}{L_4}+\arctan\frac{1}{L_5}=\arctan\frac{L_6}{L_4^2+L_3^2+L_2^2+2}$$

$$\arctan\frac{1}{L_5}+\arctan\frac{1}{L_6}=\arctan\frac{L_7}{L_5^2+L_4^2+L_3^2+L_2^2+2}$$

………

3．推廣到遞迴數列<$E_i$>： $E_1=1,E_2=n,E_{i+2}=E_i+E_{i+1},i\geq 1,i\in N$ ，其前十項如下表所示。

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $E_i$ | 1 | n | 1+n | 1+2n | 2+3n | 3+5n | 5+8n | 8+13n | 13+21n | 21+34n |

$$\arctan\frac{1}{E_1}+\arctan\frac{1}{E_2}=\arctan\frac{E_3}{E_1E_2-1}=\arctan\frac{E_3}{n-1}$$
$$\arctan\frac{1}{E_2}+\arctan\frac{1}{E_3}=\arctan\frac{E_4}{E_2^2+n-1}$$
$$\arctan\frac{1}{E_3}+\arctan\frac{1}{E_4}=\arctan\frac{E_5}{E_3^2+E_2^2+n-1}$$
$$\arctan\frac{1}{E_4}+\arctan\frac{1}{E_5}=\arctan\frac{E_6}{E_4^2+E_3^2+E_2^2+n-1}$$
$$\arctan\frac{1}{E_5}+\arctan\frac{1}{E_6}=\arctan\frac{E_7}{E_5^2+E_4^2+E_3^2+E_2^2+n-1}$$
………

4・推廣遞迴數列 H(2,i) <Hi>： $H_1=2, H_2=n, H_{i+2}=H_i+H_{i+1}, i\ge 1, i\in N$

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Hi | 2 | n | 2+n | 2+2n | 4+3n | 6+5n | 10+8n | 16+23n | 26+31n | 42+54n |

$$\arctan\frac{1}{H_1}+\arctan\frac{1}{H_2}=\arctan\frac{H_3}{H_1H_2-1}=\arctan\frac{H_3}{2n-1}$$
$$\arctan\frac{1}{H_2}+\arctan\frac{1}{H_3}=\arctan\frac{H_4}{H_2^2+2n-1}$$
$$\arctan\frac{1}{H_3}+\arctan\frac{1}{H_4}=\arctan\frac{H_5}{H_3^2+H_2^2+2n-1}$$
$$\arctan\frac{1}{H_4}+\arctan\frac{1}{H_5}=\arctan\frac{H_6}{H_4^2+H_3^2+H_2^2+2n-1}$$
$$\arctan\frac{1}{H_5}+\arctan\frac{1}{H_6}=\arctan\frac{H_7}{H_5^2+H_4^2+H_3^2+H_2^2+2n-1}$$

在第(3)中發現先前的猜測“等式中會和數列第二項和第一項的差($D_2-D_1$)有關”並不完全正確，應修正爲“和($D_1D_2$-1)”有關。

於是我們可將其歸納到所有的二階線性數列，並整理爲一般式。

定義遞迴數列<Gi>： $G_1=a, G_2=b, G_{i+2}=G_i+G_{i+1}, i\ge 1, i\in N$ 。其前十項如下表所列：

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Gi | a | b | a+b | a+2b | 2a+3b | 3a+5b | 5a+8b | 8a+13b | 13a+21b | 21a+34b |

我們發現

$$\frac{\frac{1}{a}+\frac{1}{b}}{1-\frac{1}{a}\times\frac{1}{b}}=\frac{a+b}{ab-1}$$

$$\frac{\frac{1}{b}+\frac{1}{a+b}}{1-\frac{1}{b}\times\frac{1}{a+b}}=\frac{a+2b}{b^2+ab-1}$$

$$\frac{\frac{1}{a+b}+\frac{1}{a+2b}}{1-\frac{1}{a+b}\times\frac{1}{a+2b}}=\frac{2a+3b}{(a+b)^2+b^2+ab-1}$$

$$\frac{\frac{1}{a+2b}+\frac{1}{2a+3b}}{1-\frac{1}{a+2b}\times\frac{1}{2a+3b}}=\frac{3a+5b}{(a+2b)^2+(a+b)^2+b^2+ab-1}$$

………

可整理爲:

**定理 5：**

$$\arctan\frac{1}{G_1}+\arctan\frac{1}{G_2}=\arctan\frac{G_3}{G_1G_2-1} \quad (1)$$

$$\arctan\frac{1}{G_i}+\arctan\frac{1}{G_{i+1}}=\arctan\frac{G_{i+2}}{\sum_{k=2}^{i}G_k^2+(G_1G_2-1)} \quad (2)$$

$i\geq 2$

定理五的證明：

首先要由二階線性遞迴數列一般式之平方和公式：$\sum_{k=1}^{i}G_k^2=G_iG_{i+1}+G_1^2-G_1G_2$ ， $i\in N$ [2]

取ｋ＝２時的狀況作爲引理 5.1

$$\sum_{k=2}^{i}G_k^2=\left(\sum_{k=1}^{i}G_k^2\right)-G_1^2=G_iG_{i+1}+G_1^2-G_1G_2-G_1^2=G_iG_{i+1}-G_1G_2\Leftrightarrow G_iG_{i+1}=\sum_{k=2}^{i}G_k^2+G_1G_2$$

引理 5.1:

$$G_iG_{i+1}=\sum_{k=2}^{i}G_k^2+G_1G_2$$

$$i\in N$$

第一式的證明：

$$右式=\tan\left(\arctan\frac{1}{G_1}+\arctan\frac{1}{G_2}\right)=\frac{\tan\left(\arctan\frac{1}{G_1}\right)+\tan\left(\arctan\frac{1}{G_2}\right)}{1-\tan\left(\arctan\frac{1}{G_1}\right)\tan\left(\arctan\frac{1}{G_2}\right)}$$

$$=\frac{\frac{1}{G_1}+\frac{1}{G_2}}{1-\frac{1}{G_1}\times\frac{1}{G_2}}=\frac{G_1+G_2}{G_1G_2-1}=\frac{G_3}{G_1G_2-1}$$

$$左式=\tan\left(\arctan\frac{G_3}{G_1G_2-1}\right)=\frac{G_3}{G_1G_2-1}$$

$\because$ 左式 = 右式

$\therefore$ 得證

第二式的證明：

$$右式=\tan\left(\arctan\frac{1}{G_i}+\arctan\frac{1}{G_{i+1}}\right)=\frac{\tan\left(\arctan\frac{1}{G_i}\right)+\tan\left(\arctan\frac{1}{G_{i+1}}\right)}{1-\tan\left(\arctan\frac{1}{G_i}\right)\tan\left(\arctan\frac{1}{G_{i+1}}\right)}$$

$$=\frac{\frac{1}{G_i}+\frac{1}{G_{i+1}}}{1-\frac{1}{G_i}\times\frac{1}{G_{i+1}}}=\frac{G_i+G_{i+1}}{G_iG_{i+1}-1}=\frac{G_{i+2}}{G_iG_{i+1}-1}=\frac{G_{i+2}}{\sum_{k=2}^{i}G_k^2+G_1G_2-1}$$

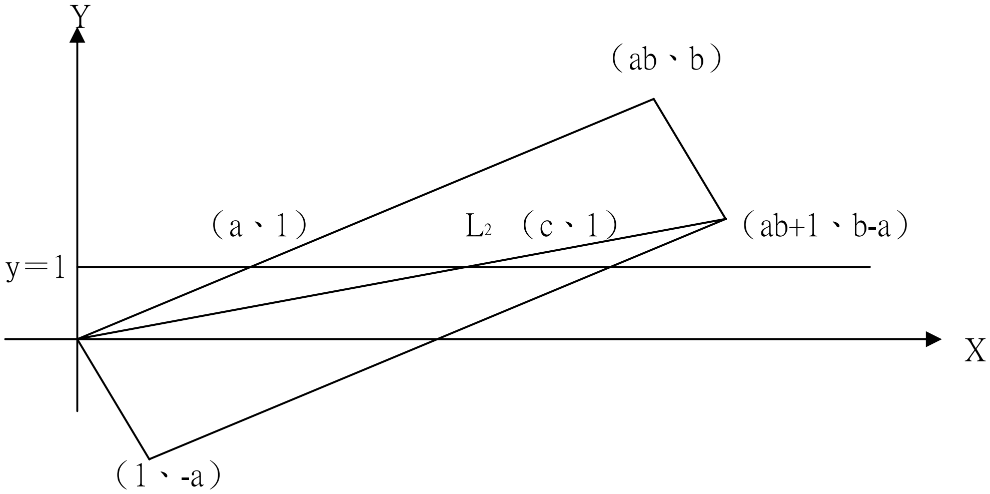
$$左式=\arctan\left(\arctan\frac{G_{i+2}}{\sum_{k=2}^{i}G_k^2+(G_1G_2-1)}\right)=\frac{G_{i+2}}{\sum_{k=2}^{i}G_k^2+G_1G_2-1}$$

∵左式＝右式

∴得證

5・探討一般式 $\tan^{-1}\frac{1}{b}+\tan^{-1}\frac{1}{c}=\tan^{-1}\frac{1}{a}$ 的正整數解

爲了解在何種條件下方格紙的疊合才具有意義，在這我們先抽出定理一和其圖形來做探討。由下圖可知：




直線 $L_2$：$y=\frac{b-a}{ab+1}x$ 與直線 y=1 相交於點(c, 1), 而 $c=\frac{ab+1}{b-a}$,

若(a,b,c)有正整數解, 則下列等式

$$\tan^{-1}\frac{1}{\mathbf{b}}+\tan^{-1}\frac{1}{\mathbf{c}}=\tan^{-1}\frac{1}{\mathbf{a}} \qquad (*)$$

也必有正整數解。

而(＊)即爲定理一的一般式，故只要求出 (a,b,c) 的正整數解，就可以知道圖形在方格紙上的疊合關係。於是我們設計了一個程式（附件一）來求出(＊)的１００組正整數解（附件二）並且整理其規律性（如下表）發現除了已知和費波那契數列有關的一組解之外尚有兩組新的解，我們將其整理爲定理六及定理七。

| a | b | c |
|---|---|---|
| i | i+1 | $i^2$+i+1 |
| 2i+1 | 2i+3 | $2i^2$+4i+2 |
| $F_{2i}$ | $F_{2i+1}$ | $F_{2i+2}$ |

定理六： $\tan^{-1}\frac{1}{i}=\tan^{-1}\frac{1}{i+1}+\tan^{-1}\frac{1}{i^2+i+1}$ ， $i\in N$

定理七： $\tan^{-1}\frac{1}{2i+1}=\tan^{-1}\frac{1}{2i+3}+\tan^{-1}\frac{1}{2i^2+4i+2}$ ， $i\in N$

## 參、結論與展望：

本文乃是一篇把幾何、代數、三角與數論結合起來的模型。

首先我們運用解析幾何及三角代數證明了 Ko 所宣稱的三個等式是對的，並且將一開始和費波那契數列有關的三個等式推廣到盧卡斯數列，甚至到二階線性遞迴數列的一般式，據我們所知這等式尚未有人提出過。運用類似的方法，我們更進一部想探討，在 a,b,c 都是正整數的情況下，a,b,c 如果有 $c=\frac{ab+1}{b-a}$ 的關係，等式 $\tan^{-1}\frac{1}{b}+\tan^{-1}\frac{1}{c}=\tan^{-1}\frac{1}{a}$ 就會成立，而滿足 $c=\frac{ab+1}{b-a}$ 的整數解 $(a,b,c)$ 到底有多少種類型呢？我們得到的有

| a | b | c |
|---|---|---|
| i | i+1 | $i^2$+i+1 |
| 2i+1 | 2i+3 | $2i^2$+4i+2 |
| $F_{2i}$ | $F_{2i+1}$ | $F_{2i+2}$ |

仔細看看卻發現在我們的研究之中尚有許多值得繼續發展的部分，例如：

1．在和費波那契數列及盧卡斯數列有關的等式皆可畫出其所對應的圖形，所以我們深信在短期之內定理五的幾何模型一定做的出來。

2．回到最初的圖形，發現只要將方格紙的尺寸以及疊合的方法稍作調整，使之點與點有所對應即可有新的等式出現。

3．本文之中只針對定理一來推廣求 $\tan^{-1}\frac{1}{b}+\tan^{-1}\frac{1}{c}=\tan^{-1}\frac{1}{a}$ 的解，相信若繼續對定理二、三…來探討必可有讓人驚訝的結果。

4．本文之中所討論的等式皆是〝兩個角合成一個角〞的形式，但只要再稍作變化便可將等式推廣到〝三個角合成一個角〞甚至是〝多個角合成多個角〞的形式。

5．由於 tangent 與 cotangent 互為倒數關係，所以和 $\cot^{-1}$ 有關的等式必和本文中所提的等式平行存在。

## 肆、參考資料


[1] KO HAYASHI(2003). Fibonacci Numbers and the Arctangent Function, Mathematics Magazine, vol.76, Iss 3 , P.214

[2] http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibFormulae.html#order2luc

附件1

graphics = 88, 116, 520, 462, , 66, 87, 498, 433, C
pset = 110, 145, 537, 491, Z, 88, 116, 515, 462, C

```
VERSION 5.00
Begin VB.Form Form1
   AutoRedraw      =   -1  'True
   Caption         =   "Form1"
   ClientHeight    =   4545
   ClientLeft      =   60
   ClientTop       =   450
   ClientWidth     =   6630
   LinkTopic       =   "Form1"
   ScaleHeight     =   5
   ScaleMode       =   0  '使用者自訂
   ScaleWidth      =   5
End
Attribute VB_Name = "Form1"
Attribute VB_GlobalNameSpace = False
Attribute VB_Creatable = False
Attribute VB_PredeclaredId = True
Attribute VB_Exposed = False
Dim a, b As Integer

Dim c, j As Integer

Dim arr(100, 100) As String
Private Sub Form_Activate()

     For a = 1 To 100
        For b = 1 To 100
           If (b - a) > 0 Then
              c = (a * b + 1) / (b - a)

              If Int(c) = c Then
                 arr(b, a) = "O"
              Else
                 arr(b, a) = "X"
              End If
```

```
            Else
               arr(b, a) = "X"
            End If

         Next
     Next

     Print Space(2); Str(1);
     For i = 2 To 100
         Print Space(1); Str(i);
     Next

     Print

     For b = 1 To 100
         Print Space(Int(b / 10)); Str(b);

         For a = 1 To 100
           Print Space(2); arr(b, a);
            'If arr(b, a) = "X" Then
            '   PSet (b, a), RGB(255, 0, 0)
            'End If
         Next
         Print
     Next

Form2.Show

End Sub
```

---

```
VERSION 5.00
Begin VB.Form graphics
   Caption         =   "graphics"
   ClientHeight    =   6420
   ClientLeft      =   60
   ClientTop       =   450
   ClientWidth     =   8115
   LinkTopic       =   "Form2"
   ScaleHeight     =   2
```

```
   ScaleMode       =   0  '使用者自訂
   ScaleWidth      =   2
   StartUpPosition =   3  '系統預設值
   Begin VB.CommandButton Command1
      Caption         =   "關閉"
      Height          =   615
      Left            =   4200
      TabIndex        =   0
      Top             =   5640
      Width           =   1335
   End
End
Attribute VB_Name = "graphics"
Attribute VB_GlobalNameSpace = False
Attribute VB_Creatable = False
Attribute VB_PredeclaredId = True
Attribute VB_Exposed = False
Dim a, b As Integer

Dim c, j As Integer

Dim arr(100, 100) As Integer
Private Sub Form_Activate()

     For a = 1 To 100
        For b = 1 To 100
           If (b - a) > 0 Then
              c = (a * b + 1) / (b - a)

              If Int(c) = c Then
                 arr(b, a) = 1
              Else
                 arr(b, a) = 0
              End If

           Else
              arr(b, a) = 0
           End If

        Next
    Next
```

```
End Sub

Private Sub Form_Paint()
    ScaleMode = 6

     For b = 1 To 100
        For a = 1 To 100
           If arr(b, a) = 1 Then
              PSet (3 * b, 3 * a), RGB(255, 0, 0)
           End If
        Next
        Print
    Next
End Sub

Private Sub Command1_Click()
    Me.Hide
End Sub
```

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 21 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 23 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 25 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 26 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 27 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 29 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 30 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 31 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 34 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 43 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 44 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 46 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 47 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 48 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 49 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 50 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 51 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 52 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 53 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 54 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 55 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 56 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 57 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 58 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 59 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 61 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 62 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 63 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 65 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 66 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 67 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 68 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 69 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 70 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 71 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 72 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 73 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 74 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 75 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 76 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 77 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 78 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 79 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 80 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 81 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 82 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 83 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 84 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 85 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 86 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 87 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 88 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 89 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 91 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 92 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 93 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 94 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 95 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 96 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 97 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 98 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 99 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 100 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

# Trigonometric Identities at the Intersection of Geometry, Algebra, Number theory, and Recursion.


**Abstract:**

This paper starts with three equations of the arc-cotangent function and Fibonacci sequences in Ko Hayashi's paper, which are illustrated without words by diagrams for two initial values. First we manage to prove the three equations by inductive reasoning and analytic geometry. Then we go on to find out similar equations of Lucas numbers and second-order liner recursive sequences as follows.

***Theorem 4:***

$$\cot^{-1} L_{2n} + \cot^{-1} L_{2n+1} = \cot^{-1} \frac{L_{4n+1}}{L_{2n+2}}$$

$$\cot^{-1} L_{2n-1} + \cot^{-1} L_{2n} = \cot^{-1} \frac{L_{4n-1} - 2}{L_{2n+1}}$$

$$n \in N$$

***Theorem 5:***

$$\cot^{-1} G_1 + \cot^{-1} G_2 = \cot^{-1} \frac{G_1 G_2 - 1}{G_3}$$

$$\cot^{-1} G_n + \cot^{-1} G_{n+1} = \cot^{-1} \frac{\sum_{k=2}^{n} G_k^2 + G_1 G_2 - 1}{G_{n+2}},\ n \geq 2\ ,\ n \in N$$

***Theorem* 6.**

$$\cot^{-1} F_n + \cot^{-1} L_n = \cot^{-1} \frac{F_{2n} - 1}{2F_{n+1}}$$

***Theorem* 7.**

$$\cot^{-1} F_{2n} + \cot^{-1} \frac{L_{4n+3} - 2}{L_{2n+3}} = \cot^{-1} \frac{F_{4n+2} - 1}{2F_{2n+2}} + \cot^{-1} \frac{F_{4n+4} - 1}{2F_{2n+3}}$$

In the end, we deal with the positive integer solutions of $ab+ac+1=bc$, $ac+2ab+4=bc$, and $ab+2ac+2=bc$, which is extended from the three equations of Fibonacci numbers. We design a computer program to figure out the solutions as follows.

| a | b | c |
|---|---|---|
| $F_{2n}$ | $F_{2n+1}$ | $F_{2n+2}$ |
| $n$ | $n+1$ | $n^2+n+1$ |
| $2n+1$ | $2n+3$ | $2n^2+4n+2$ |
| $F_{2n+2}/2$ | $F_{2n+1}$ | $F_{2n+4}$ |
| $n$ | $n+1$ | $2n^2+2n+4$ |
| $n$ | $n+2$ | $n^2+2n+2$ |
| $F_{2n}$ | $F_{2n+1}/2$ | $F_{2n+3}$ |
| $n$ | $2n+1$ | $2n^2+n+2$ |
| $n$ | $2n+2$ | $n^2+n+1$ |

## I. Motivation

During last summer vacation we read a short math paper by Ko Hayashi, who presents three equations of Fibonacci numbers and the Arc-cotangent function:

$$\cot^{-1} F_{2n} = \cot^{-1} F_{2n+1} + \cot^{-1} F_{2n+2},\ n \in N \tag{1}$$

$$\cot^{-1} \frac{F_{2n+2}}{2} = \cot^{-1} F_{2n+1} + \cot^{-1} F_{2n+4},\ n \in N \tag{2}$$

$$\cot^{-1} F_{2n} = \cot^{-1} \frac{F_{2n+1}}{2} + \cot^{-1} F_{2n+3},\ n \in N \tag{3}$$

The Fibonacci sequence $\{f_n\}_{n\geq 1} = \{1,1,2,3,5,8,13,21,\cdots\}$

Hayashi illustrates the three equations respectively with diagrams for two initial values 1 and 2. That is, there are six diagram proofs without words. For example, the following diagram, the first of the six, illustrates the first equation for $n=1$.

$\cot^{-1} 2 + \cot^{-1} 3 = \cot^{-1} 1$

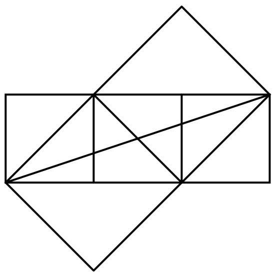


***Fig 1.*** ***Proof without words of*** $\cot^{-1} 2 + \cot^{-1} 3 = \cot^{-1} 1$

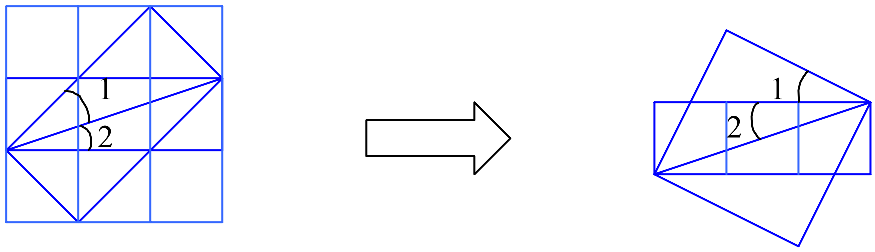
Hayashi states that only the first equation is evident in the literature. The second and third equations seem to be totally new.

Hayashi's short paper fascinated us. The combination of Fibonacci numbers and the arc-cotangent function was amazing enough, let alone the beautiful overlapping grids that served as proofs without words! However, after careful study, we began to raise questions that Hayashi's short paper failed to answer. The special cases for two initial values did not guarantee the rightness for the following values. Also, no formal proofs were provided for the second and third equations. Hence, our research work was activated.

In order to prove Hayashi's identities, we apply algebraic techniques and analytic geometry, trying to deal with the problem from a different perspective.

First, we find a new way of overlapping the grids:

(1) 3×3




Proof:

$\cot^{-1} 2 = \angle 1$

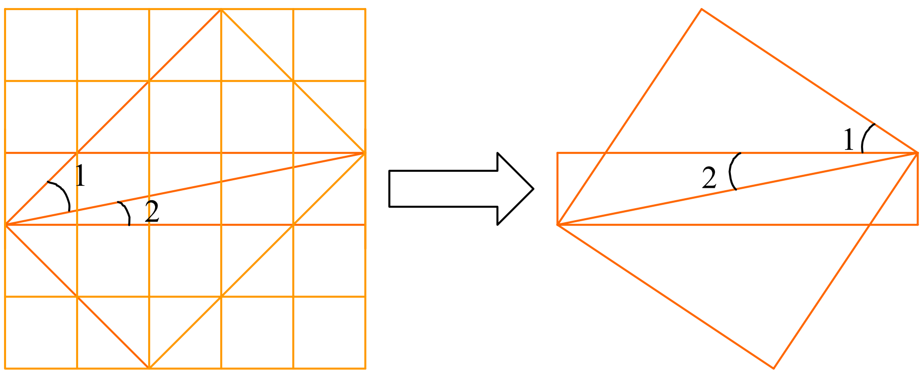
$\cot^{-1} 3 = \angle 2$

$\angle 1 + \angle 2 = 45^0$

$\cot^{-1} 1 = 45^0$

$\therefore \cot^{-1} 2 + \cot^{-1} 3 = \cot^{-1} 1$

(2) 5×5




Proof:

$\cot^{-1} \dfrac{3}{2} = \angle 1$

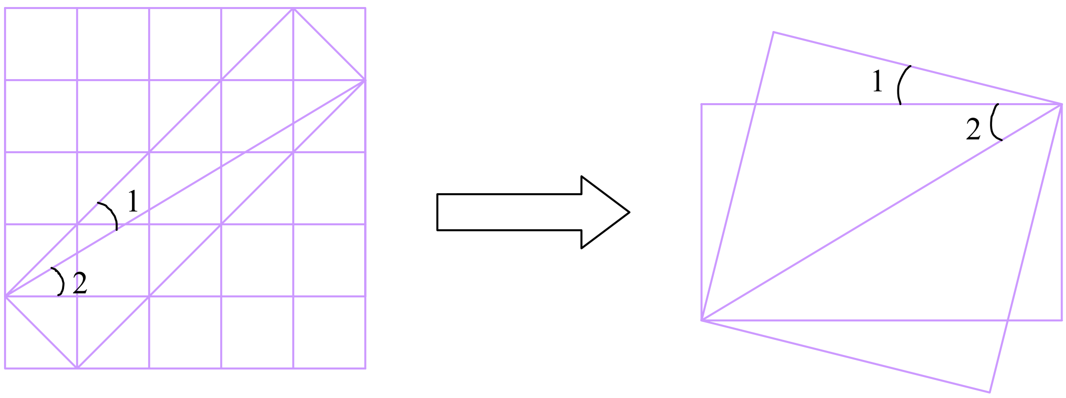
$\cot^{-1} 5 = \angle 2$

$\angle 1 + \angle 2 = 45^0$

$\cot^{-1} 1 = 45^0$

$\therefore \cot^{-1} 1 = \cot^{-1} \dfrac{3}{2} + \cot^{-1} 5$

（3） 5×5




Proof:

$\cot^{-1}4=\angle 1$

$\cot^{-1}\dfrac{5}{3}=\angle 2$

$\angle 1+\angle 2=45^{0}$

$\cot^{-1}1=45^{0}$

$\therefore \cot^{-1}1=\cot^{-1}4+\cot^{-1}\dfrac{5}{3}$

（2） $(a+b)\times(a+b)$

Proof:

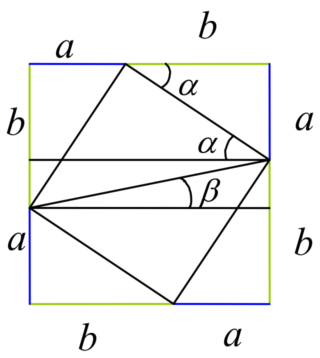



***Fig 2. Proof without words of*** $\cot^{-1}\dfrac{b}{a}+\cot^{-1}\dfrac{b+a}{b-a}=\cot^{-1}1$

$\cot\alpha=\dfrac{b}{a}\rightarrow\alpha=\cot^{-1}\dfrac{b}{a}$

$\cot\beta=\dfrac{b+a}{b-a}\rightarrow\beta=\cot^{-1}\dfrac{b+a}{b-a}$

$\tan\left(\alpha+\beta\right)=1\rightarrow\alpha+\beta=\cot^{-1}1$

$\therefore \cot^{-1}\dfrac{b}{a}+\cot^{-1}\dfrac{b+a}{b-a}=\cot^{-1}1$

A major assumption of ours is: if Hayashi's three identities are evident, they shouldn't be isolated cases. Since Fibonacci numbers and Lucas numbers share similarities, there must be similar equations of Lucas numbers if the three equations of Fibonacci numbers are evident. In addition, since Fibonacci numbers and Lucas numbers are basic second-order liner recursive sequences, there probably exist similar equations of general second-order liner recursive sequences. What's more, if equations of Fibonacci numbers can be illustrated by diagrams of overlapping grids, equations of both Lucas numbers and general second-order liner recursive sequences can also be illustrated the same way.

Our research is divided into three parts. For the beginning part, we formally prove Hayashi's three equations of Fibonacci numbers and the arc-cotangent function, accompanied by diagrams of overlapping grids. In the second part, we generalize equations of Fibonacci numbers to those of Lucas numbers, and hence obtain new equations of Lucas numbers and the arc-cotangent function. For part three, we go on to create new equations of general second-liner recursive sequences and the arc-cotangent function.

## II. Methods, Process, & Results

**Part 1: Equations of Fibonacci numbers and the arc-cotangent function & the overlapping grids**

> ***Theorem 1:***
>
> $$\cot^{-1} F_{2n} = \cot^{-1} F_{2n+1} + \cot^{-1} F_{2n+2},\ n \in N$$

**Algebraic proof of Theorem 1:**

1. We have $\cot^{-1} F_2 = \cot^{-1} F_3 + \cot^{-1} F_4$ as $n = 1$

2. Suppose that $\cot^{-1} F_{2k} = \cot^{-1} F_{2k+1} + \cot^{-1} F_{2k+2}$,

   Then we obtain that $F_{2k} = \dfrac{F_{2k+1}F_{2k+2} - 1}{F_{2k+2} + F_{2k+1}}$

   $$\therefore 1 = F_{2k+1}F_{2k+2} - F_{2k+2}F_{2k} - F_{2k+1}F_{2k}$$

3. Let $n = k + 1$. We have

$$\begin{aligned}
&F_{2k+4}F_{2k+3} - F_{2k+4}F_{2k+2} - F_{2k+3}F_{2k+2} \\
&= (F_{2k+3} + F_{2k+2})F_{2k+3} - (F_{2k+3} + F_{2k+2})(F_{2k+3} - F_{2k+1}) - F_{2k+3}(F_{2k+1} + F_{2k}) \\
&= {F_{2k+3}}^2 + F_{2k+3}F_{2k+2} - {F_{2k+3}}^2 - F_{2k+3}F_{2k+2} + F_{2k+3}F_{2k+1} + F_{2k+2}F_{2k+1} - F_{2k+3}F_{2k+1} - F_{2k+3}F_{2k} \\
&= F_{2k+2}F_{2k+1} - (F_{2k+2} + F_{2k+1})F_{2k} \\
&= F_{2k+2}F_{2k+1} - F_{2k+2}F_{2k} - F_{2k+1}F_{2k} = 1,
\end{aligned}$$

$$\therefore F_{2k+2} = \frac{F_{2k+4}F_{2k+3} - 1}{F_{2k+3} + F_{2k+4}}$$

$$\Rightarrow \cot^{-1} F_{2k} = \cot^{-1} F_{2k+1} + \cot^{-1} F_{2k+2}$$

Therefore $\cot^{-1} F_{2k} = \cot^{-1} F_{2k+1} + \cot^{-1} F_{2k+2}$ as $n = k + 1$.

**Geometry proof of Theorem 1:**

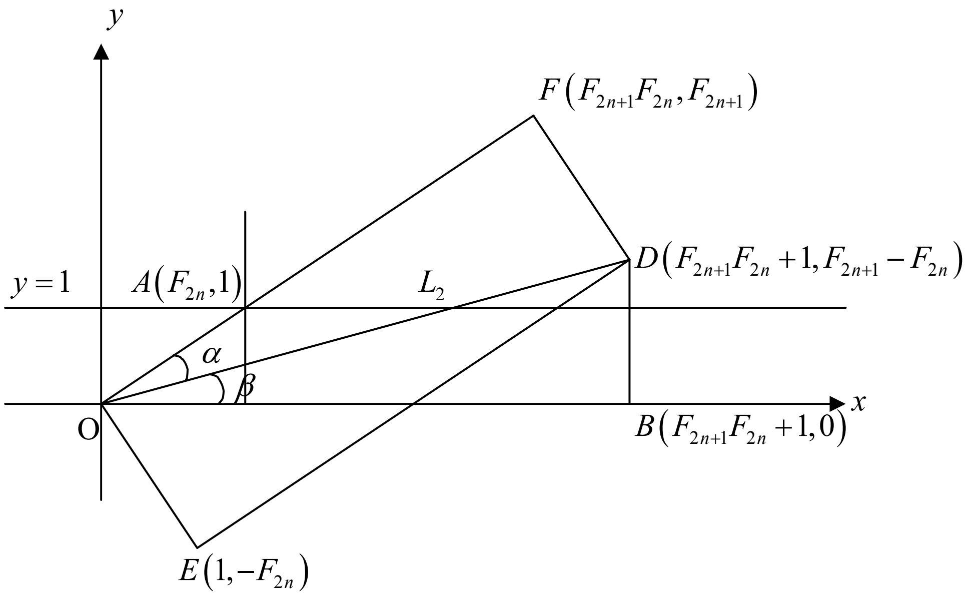



***Fig 3. Illustration of Theorem 1***

Let $\angle FOD=\alpha$, $\angle DOB=\beta$

We obtain $L_2: y=\dfrac{F_{2n+1}-F_{2n}}{F_{2n+1}F_{2n}+1}x$, and $\cot(\alpha+\beta)=F_{2n}$.

Hence $\alpha+\beta=\cot^{-1}F_{2n}$.

Similarly, in $\Delta ODF\left(\Delta OBD\right)$, we have

$$\cot\alpha=\frac{F_{2n+1}-\sqrt{F_{2n}^2+1}}{\sqrt{F_{2n}^2+1}}=F_{2n+1},\quad \cot\beta=\frac{F_{2n+1}F_{2n}+1}{F_{2n+1}-F_{2n}}=F_{2n+2}.$$

Hence $\alpha=\cot^{-1}F_{2n+1}$, $\beta=\cot^{-1}F_{2n+2}$.

Combining these formulae, we have $\cot^{-1}F_{2n}=\cot^{-1}F_{2n+1}+\cot^{-1}F_{2n+2}$. □

**Theorem 2:**

$$\cot^{-1}\frac{F_{2n+2}}{2}=\cot^{-1}F_{2n+1}+\cot^{-1}F_{2n+4},\ n\in N$$

**Algebraic proof of Theorem 2:**

1.We have $\cot^{-1}\frac{F_4}{2}=\cot^{-1}F_3+\cot^{-1}F_6$ as $n=1$

2.Suppose that $\cot^{-1}\frac{F_{2k+2}}{2}=\cot^{-1}F_{2k+1}+\cot^{-1}F_{2k+4}$,

than we obtain $\frac{F_{2k+2}}{2}=\frac{F_{2k+4}F_{2k+1}-1}{F_{2k+1}+F_{2k+4}}$

$$2F_{2k+4}F_{2k+1}-2=F_{2k+2}F_{2k+1}+F_{2k+2}F_{2k+4}$$

$$2F_{2k+4}F_{2k+1}-F_{2k+2}F_{2k+1}-F_{2k+2}F_{2k+4}=2$$

3.Let $n=k+1$. We have

$$\begin{aligned}
&2F_{2k+6}F_{2k+3}-F_{2k+4}F_{2k+3}-F_{2k+4}F_{2k+6}\\
&=2(F_{2k+5}+F_{2k+4})F_{2k+3}-(F_{2k+5}F_{2k+4})F_{2k+4}-F_{2k+4}F_{2k+3}\\
&=2F_{2k+3}(2F_{2k+4}+F_{2k+3})-(2F_{2k+4}+F_{2k+3})F_{2k+4}-F_{2k+4}F_{2k+3}\\
&=4F_{2k+3}F_{2k+4}+2F_{2k+3}{}^2-2F_{2k+4}{}^2-2F_{2k+4}F_{2k+3}\\
&=2F_{2k+4}F_{2k+2}+2F_{2k+4}F_{2k+1}+2F_{2k+3}{}^2-2F_{2k+3}{}^2-4F_{2k+3}F_{2k+2}-2F_{2k+2}{}^2\\
&=2F_{2k+4}F_{2k+1}+2F_{2k+4}F_{2k+2}-2F_{2k+3}F_{2k+2}-2F_{2k+3}F_{2k+2}-F_{2k+2}{}^2\\
&=-2F_{2k+3}F_{2k+2}+2F_{2k+4}F_{2k+1}\\
&=2F_{2k+4}F_{2k+1}-2(F_{2k+4}-F_{2k+2})F_{2k+2}\\
&=2F_{2k+4}F_{2k+1}-2F_{2k+4}F_{2k+2}+2F_{2k+2}{}^2\\
&=2F_{2K+4}F_{2K+1}-F_{2k+2}F_{2k+4}-F_{2k+2}F_{2k+4}+2F_{2k+2}{}^2\\
&=2F_{2k+4}F_{2k+1}-F_{2k+2}F_{2k+4}-F_{2k+2}(2F_{2k+2}+F_{2k+1})+2F_{2k+2}{}^2\\
&=2F_{2k+4}F_{2k+1}-F_{2k+2}F_{2k+4}-2F_{2k+2}{}^2-F_{2k+2}F_{2k+1}+2F_{2k+2}{}^2\quad=2
\end{aligned}$$

Therefore $\cot^{-1}\frac{F_{2k+2}}{2}=\cot^{-1}F_{2k+1}+\cot^{-1}F_{2k+4}$ as $n=k+1$.

**Geometric proof of Theorem 2:**

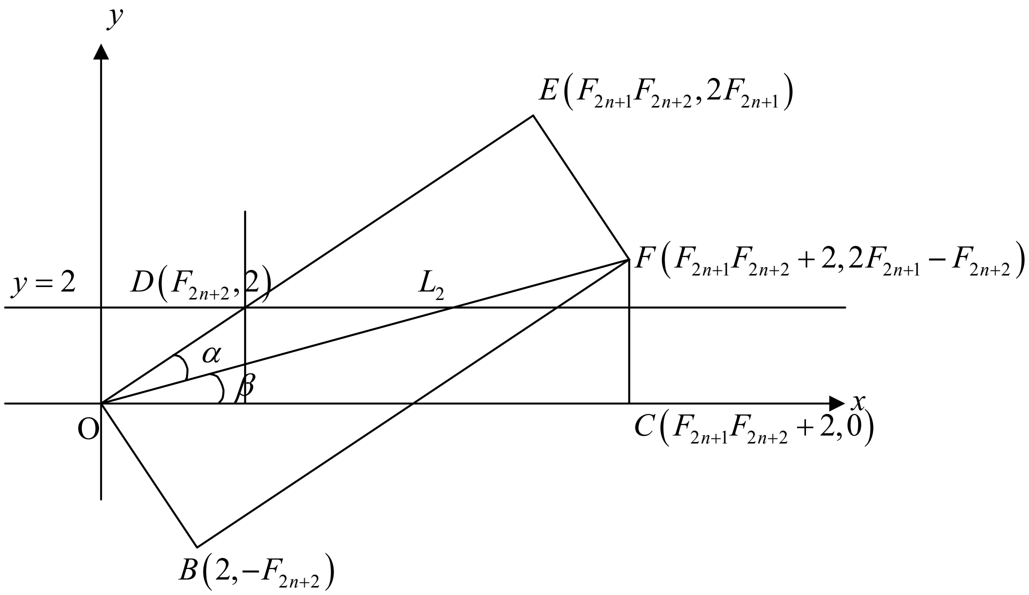



***Fig 4. Illustration of Theorem 2***

Let $\overline{OD}=K$

$$\tan(\alpha+\beta)=\frac{2}{F_{2n+2}}$$

$$\tan\alpha=\frac{K}{F_{2n+1}K}=\frac{1}{F_{2n+1}} \qquad \alpha=\tan^{-1}\frac{1}{F_{2n+1}}$$

$$\beta=\tan^{-1}\frac{2F_{2n+1}-F_{2n+2}}{F_{2n+1}F_{2n+2}+2}=\tan^{-1}\frac{1}{F_{2n+4}}$$

Combining these formulae, we have $\cot^{-1}\frac{F_{2n+2}}{2}=\cot^{-1}F_{2n+1}+\cot^{-1}F_{2n+4}$. □

***Theorem 3:***

$$\cot^{-1} F_{2n} = \cot^{-1} \frac{F_{2n+1}}{2} + \cot^{-1} F_{2n+3},\ n \in N$$

**Algebraic proof of Theorem 3:**

1. We have $\cot^{-1} F_2 = \cot^{-1} \frac{F_3}{2} + \cot^{-1} F_5$ as $n = 1$

2. Suppose that $\cot^{-1} F_{2k} = \cot^{-1} \frac{F_{2k+1}}{2} + \cot^{-1} F_{2k+3}$.

   Than we obtain that $F_{2k} = \dfrac{F_{2k+2}F_{2k+3} - 2}{2F_{2k+3} + F_{2k+2}}$

$$F_{2k+3}F_{2k+2} - 2F_{2k+3}F_{2k} - F_{2k}F_{2k+2} = 2$$

3. Let $n = k+1$. We have

$$\begin{aligned}
& F_{2k+5}F_{2k+4} - 2F_{2k+5}F_{2k+2} - F_{2k+4}F_{2k+2} \\
&= (F_{2k+4} + F_{2k+3})(F_{2k+3} + F_{2k+2}) - 2F_{2k+2}(F_{2k+4} + F_{2k+3}) - F_{2k+2}(F_{2k+3} + F_{2k+2}) \\
&= F_{2k+4}F_{2k+3} + F_{2k+4}F_{2k+2} + F_{2k+3}{}^2 + F_{2k+3}F_{2k+2} - 2F_{2k+4}F_{2k+2} - 2F_{2k+2}F_{2k+3} - F_{2k+3}F_{2k+2} - F_{2k+2}{}^2 \\
&= F_{2k+4}F_{2k+3} - F_{2k+4}F_{2k+2} - 2F_{2k+2}F_{2k+3} + F_{2k+3}{}^2 - F_{2k+2}{}^2 \\
&= F_{2k+3}(F_{2k+3} + F_{2k+2}) - F_{2k+2}(F_{2k+3} + F_{2k+2}) - 2F_{2k+3}(F_{2k+1} + F_{2k}) + F_{2k+3}{}^2 - F_{2k+2}{}^2 \\
&= F_{2k+3}{}^2 + F_{2k+3}F_{2k+2} - F_{2k+3}F_{2k+2} - F_{2k+2}{}^2 - 2F_{2k+3}F_{2k+1} - 2F_{2k+3}F_{2k} + F_{2k+3}{}^2 - F_{2k+2}{}^2 - 2F_{2k+3}F_{2k} \\
&= 2(F_{2k+3}{}^2 - F_{2k+2}{}^2) - 2F_{2k+3}F_{2k+1} - 2F_{2k+3}F_{2k} \\
&= 2(2F_{2k+2}F_{2k+3} - 2F_{2k+2}{}^2 + F_{2k+3}F_{2k+1} - F_{2k+2}F_{2k+1}) - 2(F_{2k+3}F_{2k+1}) - 2F_{2k+3}F_{2k} \\
&= 4F_{2k+2}F_{2k+3} - 4_{2k+2}{}^2 - 2F_{2k+2}F_{2k+1} - 2F_{2k+3}F_{2k} \\
&= F_{2k+2}(4F_{2k+3} - 4F_{2k+2} - 2F_{2k+1}) - 2F_{2k+3}F_{2k} \\
&= F_{2k+2}[4(F_{2k+2} + F_{2k+1}) - 4F_{2k+2} - 2F_{2k+1}] - 2F_{2k+3}F_{2k} \\
&= F_{2k+2}(2F_{2k+1}) - 2F_{2k+3}F_{2k} \\
&= F_{2k+2}(F_{2k+3} - F_{2k}) - 2F_{2k+3}F_{2k} \\
&= F_{2k+3}F_{2k+1} - 2F_{2k+3}F_{2k} - F_{2k}F_{2k+1} = 2
\end{aligned}$$

Therefore $F_{2k+3}F_{2k+1} - 2F_{2k+3}F_{2k} - F_{2k}F_{2k+1}$ as $n = k+1$.

**Geometric proof of Theorem 3:**

Let $\overline{OA} = K$

$$\tan(\alpha+\beta) = \frac{1}{F_{2n}}$$

$$\tan\alpha = \frac{2K}{F_{2n+2}K} = \frac{2}{F_{2n+2}} \qquad \therefore \alpha = \tan^{-1}\frac{2}{F_{2n+2}}$$

$$\beta = \tan^{-1}\frac{F_{2n+2}-2F_{2n}}{F_{2n+2}F_{2n}+2} = \tan^{-1}\frac{1}{F_{2n+3}}$$

Combining these formulae, we have $\cot^{-1}F_{2n} = \cot^{-1}\dfrac{F_{2n+1}}{2} + \cot^{-1}F_{2n+3}$ . □

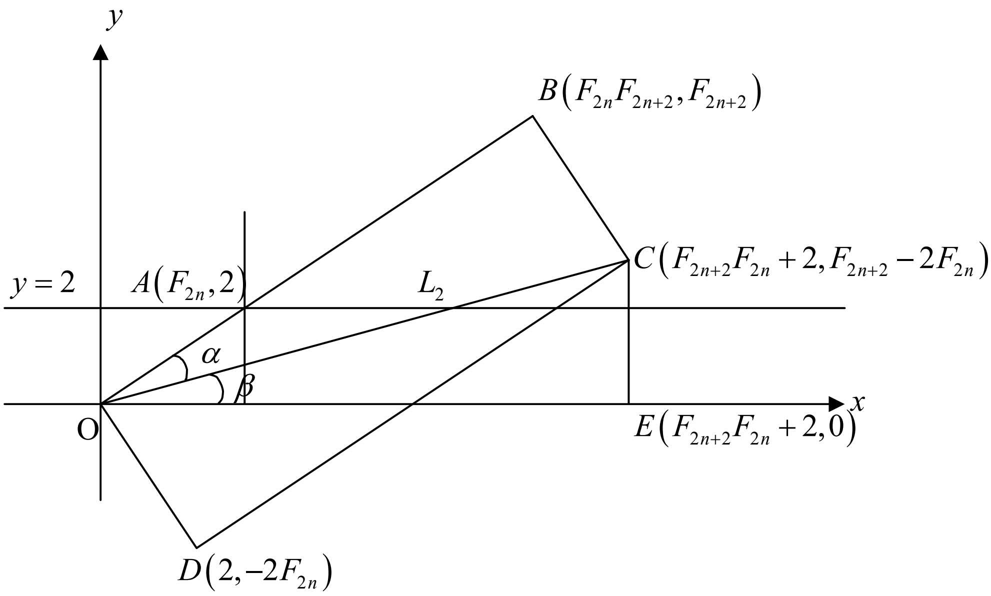



***Fig 5. Illustration of Theorem 3***

**Part 2: Equations of Lucas numbers and the arc-cotangent function & the overlapping grids**

In this part, the relation between Lucas numbers and the arc-cotangent function is established, which leads to new equations. The research methods involve trigonometry and algebra, plus properties of Lucas numbers.

The Lucas numbers are defined by $L_1 = 1, L_2 = 3, L_{n+2} = L_n + L_{n+1}, n \in N$
The ten initial values are listed in the following table.

| N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_n$ | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 |

First, we apply $\cot(\alpha+\beta) = \dfrac{\tan\beta\tan\alpha - 1}{\tan\alpha + \tan\beta}$

We make $\cot\alpha = 3, \cot\beta = 4, \cdots$, in order to figure out the relation between Lucas numbers and the arc-cotangent function. We thus obtain two new findings.

(1)

$$\cot^{-1} L_2 + \cot^{-1} L_3 = \cot^{-1} \frac{L_{2+3}}{L_4}$$

$$\cot^{-1} L_4 + \cot^{-1} L_5 = \cot^{-1} \frac{L_{4+5}}{L_6}$$

$$\cot^{-1} L_6 + \cot^{-1} L_7 = \cot^{-1} \frac{L_{6+7}}{L_8}$$

$$\cot^{-1} L_8 + \cot^{-1} L_9 = \cot^{-1} \frac{L_{8+9}}{L_{10}}$$

$$\cot^{-1} L_{10} + \cot^{-1} L_{11} = \cot^{-1} \frac{L_{10+11}}{L_{12}}$$

.........

$$Example: \cot^{-1} 3 + \cot^{-1} 4 = \cot^{-1} \frac{11}{7}$$

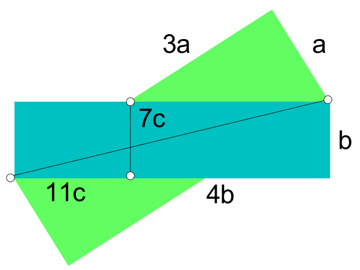





***Fig 6.*** ***Proof without words of*** $\cot^{-1} 3 + \cot^{-1} 4 = \cot^{-1} \frac{11}{7}$

(2)

$$\cot^{-1} L_1 + \cot^{-1} L_2 = \cot^{-1} \frac{L_{1+2} - 2}{L_3}$$

$$\cot^{-1} L_3 + \cot^{-1} L_4 = \cot^{-1} \frac{L_{3+4} - 2}{L_5}$$

$$\cot^{-1} L_5 + \cot^{-1} L_6 = \cot^{-1} \frac{L_{5+6} - 2}{L_7}$$

$$\cot^{-1} L_7 + \cot^{-1} L_8 = \cot^{-1} \frac{L_{7+8} - 2}{L_9}$$

$$\cot^{-1} L_9 + \cot^{-1} L_{10} = \cot^{-1} \frac{L_{9+10} - 2}{L_{11}}$$

.........

$$\mathit{Example}: \cot^{-1} 4 + \cot^{-1} 7 = \cot^{-1} \frac{27}{11}$$

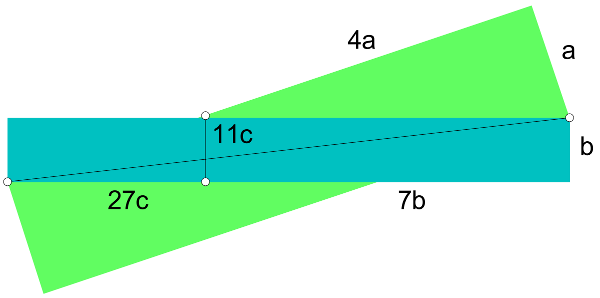



***Fig 7.*** ***Proof without words of*** $\cot^{-1} 4 + \cot^{-1} 7 = \cot^{-1} \frac{27}{11}$

Accordingly, we create a few theorems, which will later be proved by trigonometry and known properties of Lucas numbers. From the known Lucas property $L_mL_n=L_{n+m}+(-1)^mL_{n-m}$ , n＞m [2], we obtain theorem4.1:

**Lemma 4.1:**

Let $\langle L_n \rangle$ equals to Lucas numbers, $k \in N$, than

$$L_k L_{k+1} = L_{2k+1} + (-1)^k$$

***Theorem 4:***

$$\cot^{-1} L_{2n} + \cot^{-1} L_{2n+1} = \cot^{-1} \frac{L_{4n+1}}{L_{2n+2}}$$

$$\cot^{-1} L_{2n-1} + \cot^{-1} L_{2n} = \cot^{-1} \frac{L_{4n-1} - 2}{L_{2n+1}}$$

$$n \in N$$

The Proof of Theorem 4:

The first identity:

$$\cot\left(\cot^{-1} L_{2n} + \cot^{-1} L_{2n+1}\right) = \cot\left(\cot^{-1} \frac{L_{4n+1}}{L_{2n+2}}\right)$$

$$\Leftrightarrow \frac{\cot\left(\cot^{-1} L_{2n+1}\right)\cot\left(\cot^{-1} L_{2n}\right) - 1}{\cot\left(\cot^{-1} L_{2n+1}\right) + \cot\left(\cot^{-1} L_{2n}\right)} = \frac{L_{4n+1}}{L_{2n+2}}$$

$$\Leftrightarrow \frac{L_{2n+1}L_{2n} - 1}{L_{2n+1} + L_{2n}} = \frac{L_{4n+1}}{L_{2n+2}}$$

$$\Leftrightarrow \frac{L_{2n+1}L_{2n} - 1}{L_{2n+2}} = \frac{L_{4n+1}}{L_{2n+2}}$$

$$\Leftrightarrow L_{4n+1} + 1 = L_{2n}L_{2n+1}$$

Than apply *Lemma 4.1*

The second identity:

$$\cot\left(\cot^{-1}L_{2n-1}+\cot^{-1}L_{2n}\right)=\cot\left(\cot^{-1}\frac{L_{4n-1}-2}{L_{2n+1}}\right)$$

$$\Leftrightarrow\frac{\cot\left(\cot^{-1}L_{2n-1}\right)\cot\left(\cot^{-1}L_{2n}\right)-1}{\cot\left(\cot^{-1}L_{2n-1}\right)+\cot\left(\cot^{-1}L_{2n}\right)}=\frac{L_{4n-1}-2}{L_{2n+1}}$$

$$\Leftrightarrow\frac{L_{2n}+L_{2n-1}}{L_{2n}L_{2n-1}-1}=\frac{L_{2n+1}}{L_{4n-1}-2}$$

$$\Leftrightarrow\frac{L_{2n+1}}{L_{2n}L_{2n-1}-1}=\frac{L_{2n+1}}{L_{4n-1}-2}$$

$$\Leftrightarrow L_{4n-1}-1=L_{2n}L_{2n-1}$$

Than apply *Lemma 4.1*.

**Part 3: Equations of second-order liner recursive sequences and the arc-cotangent function**

Here we define a new recursive sequence $D(1,4)$ as follow:

Sequence $\langle D_n \rangle$ satisfies $D_1 = 1, D_2 = 4, D_{n+2} = D_{n+1} + D_n, n \in N$.

The ten initial values are listed in the following table.

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $D_n$ | 1 | 4 | 5 | 9 | 14 | 23 | 37 | 60 | 97 | 157 |

Something different from what we have found in the Fibonacci numbers and the Lucas numbers in observed here in the sequence $D(1,4)$ — the assumed $(D_2 - D_1)$ seems to be the answer. However, what is true with the sequence $D(1,4)$ also applies to the Fibonacci numbers and the Lucas numbers. To modify the assumption, we go on with the following new equations:

1. The new equations of the Fibonacci numbers:

| N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |

$$\cot^{-1} F_1 + \cot^{-1} F_2 = \cot^{-1} \frac{F_2 - F_1}{F_3} = \cot^{-1} \frac{0}{F_3}$$

$$\cot^{-1} F_2 + \cot^{-1} F_3 = \cot^{-1} \frac{F_2^2 + 0}{F_4}$$

$$\cot^{-1} F_3 + \cot^{-1} F_4 = \cot^{-1} \frac{F_3^2 + F_2^2}{F_5}$$

$$\cot^{-1} F_4 + \cot^{-1} F_5 = \cot^{-1} \frac{F_4^2 + F_3^2 + F_2^2}{F_6}$$

$$\cot^{-1} F_5 + \cot^{-1} F_6 = \cot^{-1} \frac{F_5^2 + F_4^2 + F_3^2 + F_2^2}{F_7}$$

………

2. The new equations of the Lucas numbers:

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_n$ | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 |

$$\cot^{-1} L_1 + \cot^{-1} L_2 = \cot^{-1}\frac{L_2 - L_1}{L_3} = \arctan\frac{2}{L_3}$$

$$\cot^{-1} L_2 + \cot^{-1} L_3 = \cot^{-1}\frac{L_2^2 + 2}{L_4}$$

$$\cot^{-1} L_3 + \cot^{-1} L_4 = \cot^{-1}\frac{L_3^2 + L_2^2 + 2}{L_5}$$

$$\cot^{-1} L_4 + \cot^{-1} L_5 = \cot^{-1}\frac{L_4^2 + L_3^2 + L_2^2 + 2}{L_6}$$

$$\cot^{-1} L_5 + \cot^{-1} L_6 = \cot^{-1}\frac{L_5^2 + L_4^2 + L_3^2 + L_2^2 + 2}{L_7}$$

………

3. The generalization to the recursive sequence
＜$E_n$＞: $E_1 = 1, E_2 = n, E_{n+2} = E_n + E_{n+1}, n \in N$.
In the following are the ten initial values.

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $E_n$ | 1 | n | 1+n | 1+2n | 2+3n | 3+5n | 5+8n | 8+13n | 13+21n | 21+34n |

$$\cot^{-1} E_1 + \cot^{-1} E_2 = \cot^{-1}\frac{E_1E_2 - 1}{E_3} = \cot^{-1}\frac{n-1}{E_3}$$

$$\cot^{-1} E_2 + \cot^{-1} E_3 = \cot^{-1}\frac{E_2^2 + n - 1}{E_4}$$

$$\cot^{-1} E_3 + \cot^{-1} E_4 = \cot^{-1}\frac{E_3^2 + E_2^2 + n - 1}{E_5}$$

$$\cot^{-1} E_4 + \cot^{-1} E_5 = \cot^{-1}\frac{E_4^2 + E_3^2 + E_2^2 + n - 1}{E_6}$$

$$\cot^{-1} E_5 + \cot^{-1} E_6 = \cot^{-1}\frac{E_5^2 + E_4^2 + E_3^2 + E_2^2 + n - 1}{E_7}$$

………

4. Further generalization H(2,n) <$H_n$>: $H_1 = 2, H_2 = n, H_{n+2} = H_n + H_{n+1}, n \in N$
In the following are the ten initial values.

| H | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $H_n$ | 2 | n | 2+n | 2+2n | 4+3n | 6+5n | 10+8n | 16+13n | 26+21n | 42+34n |

$$\cot^{-1} H_1 + \cot^{-1} H_2 = \cot^{-1} \frac{H_1 H_2 - 1}{H_3} = \cot^{-1} \frac{2n-1}{H_3}$$

$$\cot^{-1} H_2 + \cot^{-1} H_3 = \cot^{-1} \frac{H_2^2 + 2n - 1}{H_4}$$

$$\cot^{-1} H_3 + \cot^{-1} H_4 = \cot^{-1} \frac{H_3^2 + H_2^2 + 2n - 1}{H_5}$$

$$\cot^{-1} H_4 + \cot^{-1} H_5 = \cot^{-1} \frac{H_4^2 + H_3^2 + H_2^2 + 2n - 1}{H_6}$$

$$\cot^{-1} H_5 + \cot^{-1} H_6 = \cot^{-1} \frac{H_5^2 + H_4^2 + H_3^2 + H_2^2 + 2n - 1}{H_7}$$

………

Now we are confident to modify the assumed $(D_2 - D_1)$ into $D_1 D_2 - 1$, and hence create the general second-order liner recursive sequence.

5. We defined the recursive sequence
<$G_n$>: $G_1 = a, G_2 = b, G_{i+2} = G_i + G_{i+1}, i \geq 1, i \in N$.
And the ten initial values are listed as follows:

| N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $G_n$ | a | b | a+b | a+2b | 2a+3b | 3a+5b | 5a+8b | 8a+13b | 13a+21b | 21a+34b |

We fined:

$$\frac{b \times a - 1}{b + a} = \frac{ab - 1}{a + b}$$

$$\frac{(a+b) \times b - 1}{(a+b) + b} = \frac{b^2 + ab - 1}{a + 2b}$$

$$\frac{(a+2b) \times (a+b) - 1}{(a+2b) + (a+b)} = \frac{(a+b)^2 + b^2 + ab - 1}{2a + 3b}$$

………

We thus came up with Theorem 5:

***Theorem 5:***

$$\cot^{-1} G_1 + \cot^{-1} G_2 = \cot^{-1} \frac{G_1 G_2 - 1}{G_3}$$

$$\cot^{-1} G_n + \cot^{-1} G_{n+1} = \cot^{-1} \frac{\sum_{k=2}^{n} G_k^2 + G_1 G_2 - 1}{G_{n+2}}, \ n \geq 2 \ , \ n \in N$$

**Algebraic proof of Theorem 5:**

First, for Lemma 5.1, we makes k=2 in the addition formula of second-order liner recursive sequence: $\sum_{k=1}^{i} G_k^2 = G_i G_{i+1} + G_1^2 - G_1 G_2$ , $i \in N$ 【2】.

$$\sum_{k=2}^{i} G_k^2 = \left( \sum_{k=1}^{i} G_k^2 \right) - G_1^2 = G_i G_{i+1} + G_1^2 - G_1 G_2 - G_1^2 = G_i G_{i+1} - G_1 G_2 \Leftrightarrow G_i G_{i+1} = \sum_{k=2}^{i} G_k^2 + G_1 G_2$$

Lemma 5.1:

$$G_i G_{i+1} = \sum_{k=2}^{i} G_k^2 + G_1 G_2$$

$$i \in N$$

The first identity:

$$\text{left identity} = \cot\left(\cot^{-1} G_1 + \cot^{-1} G_2\right) = \frac{\cot\left(\cot^{-1} G_2\right)\cot\left(\cot^{-1} G_1\right) - 1}{\cot\left(\cot^{-1} G_2\right) + \cot\left(\cot^{-1} G_1\right)}$$

$$= \frac{G_1 G_2 - 1}{G_1 + G_2} = \frac{G_1 G_2 - 1}{G_3}$$

$$\text{right identity} = \cot\left(\cot^{-1} \frac{G_1 G_2 - 1}{G_3}\right) = \frac{G_1 G_2 - 1}{G_3}$$

The second identity:

$$\text{left identity} = \cot\left(\cot^{-1} G_n + \cot^{-1} G_{n+1}\right) = \frac{\cot\left(\cot^{-1} G_{n+1}\right) + \cot\left(\cot^{-1} G_n\right) - 1}{\cot\left(\cot^{-1} G_n\right) + \cot\left(\cot^{-1} G_{n+1}\right)}$$

$$= \frac{G_n G_{n+1} - 1}{G_n + G_{n+1}} = \frac{G_n G_{n+1} - 1}{G_{n+2}} = \frac{\sum_{k=2}^{n} G_k^2 + G_1 G_2 - 1}{G_{n+2}}$$

$$\text{right identity} = \cot\left(\cot^{-1} \frac{\sum_{k=2}^{n} G_k^2 + (G_1 G_2 - 1)}{G_{n+2}}\right) = \frac{\sum_{k=2}^{n} G_k^2 + (G_1 G_2 - 1)}{G_{n+2}}$$

**Geometric proof of Theorem 5:**

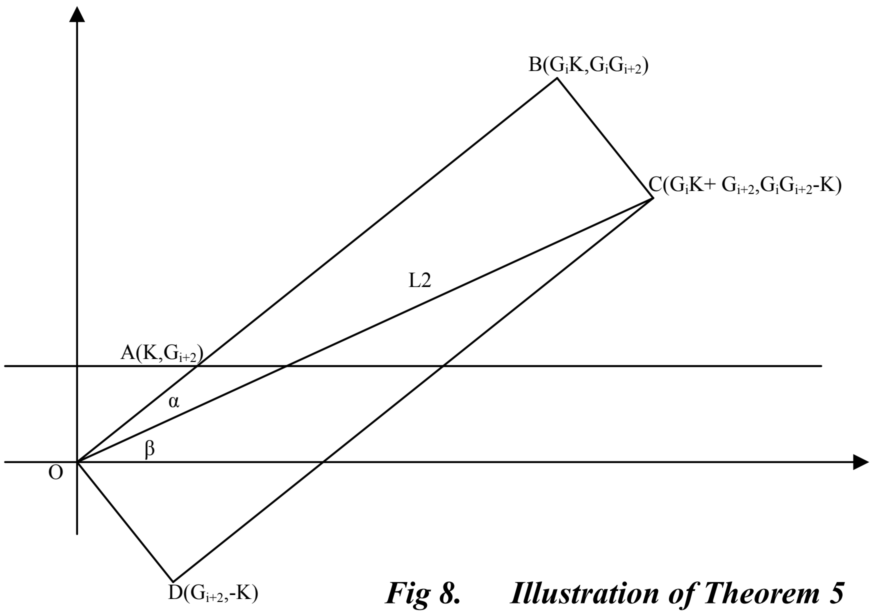



***Fig 8. Illustration of Theorem 5***

*Let* $K_n = \sum_{k=2}^{n} G_k^2 + \left(G_1 G_2 - 1\right)$, than according to *Lemma 5.1*, we get $K_n = \left(G_n G_{n+1} - 1\right)$.

$$\cot\alpha = \frac{G_n \sqrt{K_n^2 + G_{n+2}^2}}{\sqrt{K_n^2 + G_{n+2}^2}} = G_n$$

$$\cot\beta = \frac{G_n\left(G_n G_{n+1} - 1\right) + G_{n+2}}{G_n G_{n+2} - \left(G_n G_{n+1} - 1\right)} = \frac{G_n^2 G_{n+1} - G_n + \left(G_n + G_{n+1}\right)}{G_n\left(G_{n+2} - G_{n+1}\right) + 1} = \frac{G_{n+1}\left(G_n^2 + 1\right)}{G_n^2 + 1} = G_{n+1}$$

$$\cot\left(\alpha + \beta\right) = \frac{K_n}{G_{n+2}} \qquad \square$$

**Part 4: Formulae involving both the Fibonacci Numbers and Lucas Numbers**

Thus far all the identities are expressed in terms of one particular sequence of numbers. So we want to combine at least two sequences into one identity, and the following ***Theorem 6*** comes from the result after combining the Fibonacci numbers and the Lucas numbers.

***Theorem 6.***

$$\cot^{-1} F_n + \cot^{-1} L_n = \cot^{-1} \frac{F_{2n} - 1}{2F_{n+1}}$$

**Proof:** Take the cotangent on both sides of the identities and apply the Lemma 6.1 &6.2.

***Lemma 6.1:*** $F_n + L_n = 2F_{n+1}$
***Lemma 6.2:*** $F_n L_n = F_{2n}$

.□

We note that the above result is essentially based on decomposing an angle into two smaller ones. What if the angles be decomposed further? Indeed, more interesting identities are produced this way.

According to ***Theorem 1***, ***Theorem 4***, and ***Theorem 6***, we obtain a new identity as follow:

$$\left(\cot^{-1} F_{2n+1} + \cot^{-1} F_{2n+2}\right) + \left(\cot^{-1} L_{2n+1} + \cot^{-1} L_{2n+2}\right) = \cot^{-1} F_{2i} + \cot^{-1} \frac{L_{4n+3} - 2}{L_{2n+3}}$$

$$\left(\cot^{-1} F_{2n+1} + \cot^{-1} L_{2n+1}\right) + \left(\cot^{-1} F_{2n+2} + \cot^{-1} L_{2n+2}\right) = \cot^{-1} \frac{F_{4n+2} - 1}{2F_{2n+2}} + \cot^{-1} \frac{F_{4n+4} - 1}{2F_{2n+3}}$$

Then we get the ***Theorem 7***

***Theorem 7.***

$$\cot^{-1} F_{2n} + \cot^{-1} \frac{L_{4n+3} - 2}{2L_{2n+3}} = \cot^{-1} \frac{F_{4n+2} - 1}{2F_{2n+2}} + \cot^{-1} \frac{F_{4n+4} - 1}{2F_{2n+3}}$$

We remark that all our results can be doubled instantly since the tangent function and the cotangent function are reciprocal to each other. For instance, Theorem 7 can also take this form:

$$\tan^{-1}\frac{1}{F_{2n}}+\tan^{-1}\frac{2L_{2n+3}}{L_{4n+3}-2}=\tan^{-1}\frac{2F_{2n+2}}{F_{4n+2}-1}+\tan^{-1}\frac{2F_{2n+3}}{F_{4n+4}-1}$$

**Part 5: Positive Integral solutions of** $\cot^{-1} b + \cot^{-1} c = \cot^{-1} a$

For understanding that in what kinds of situations the overlapping grids are meaningful, first, we discuss diagrams generalized from ***Theorem 1***. We find that $\cot^{-1} b + \cot^{-1} c = \cot^{-1} a$ has positive integer solutions if and only if $ab + ac + 1 = bc$ has positive integer solutions.

Similarly, *Theorems 2* and *3* seek for solutions of $ac + 2ab + 4 = bc$ and $ab + 2ac + 2 = bc$.

Then we design a computer program to figure out 100 patterns of positive integer solutions, and induce its regularity (Table one). Except the pattern of Fibonacci numbers, there are still some other new patterns.

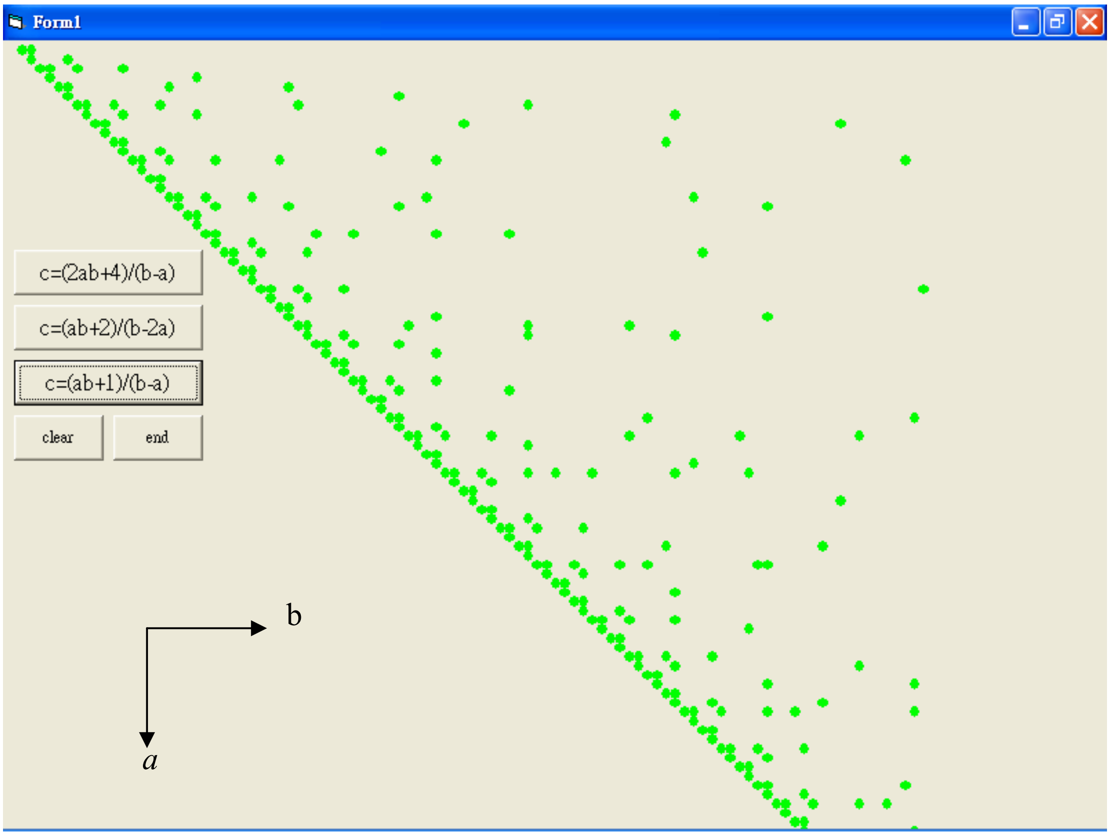



***Fig 9. Computer solutions of*** $ab + ac + 1 = bc$

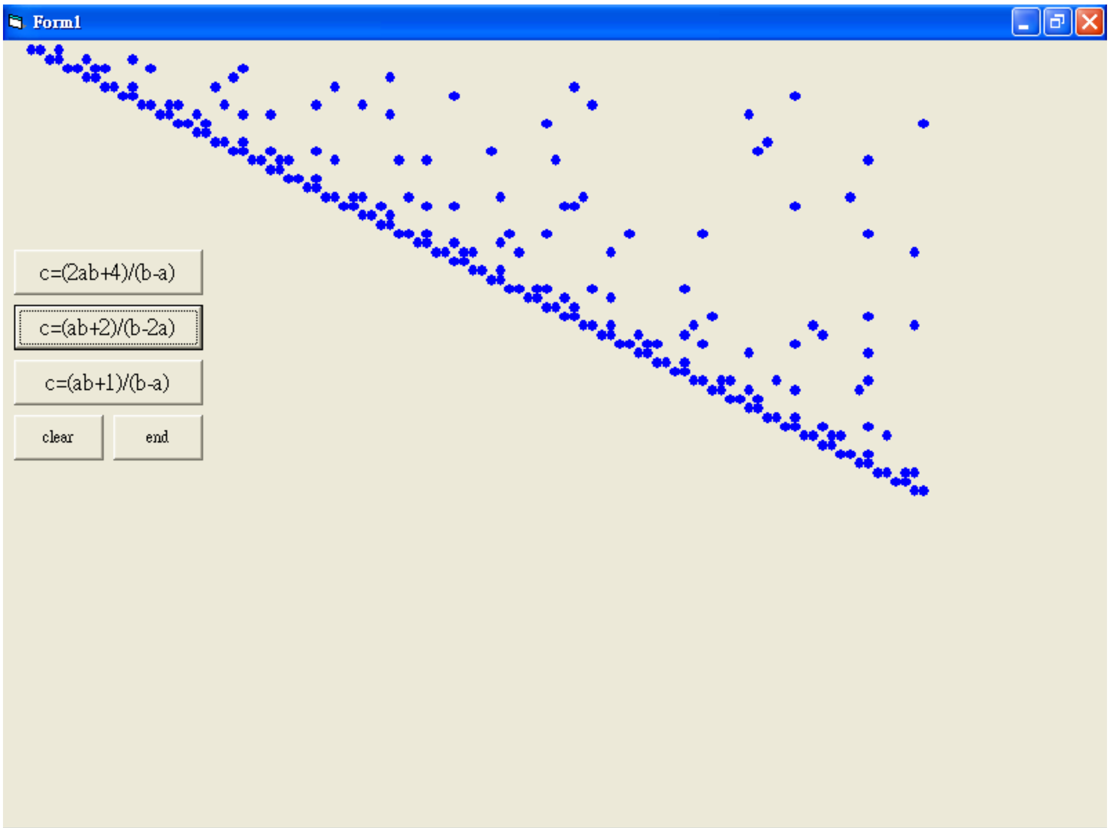



***Fig 10.*** ***Computer solutions of*** $ac + 2ab + 4 = bc$

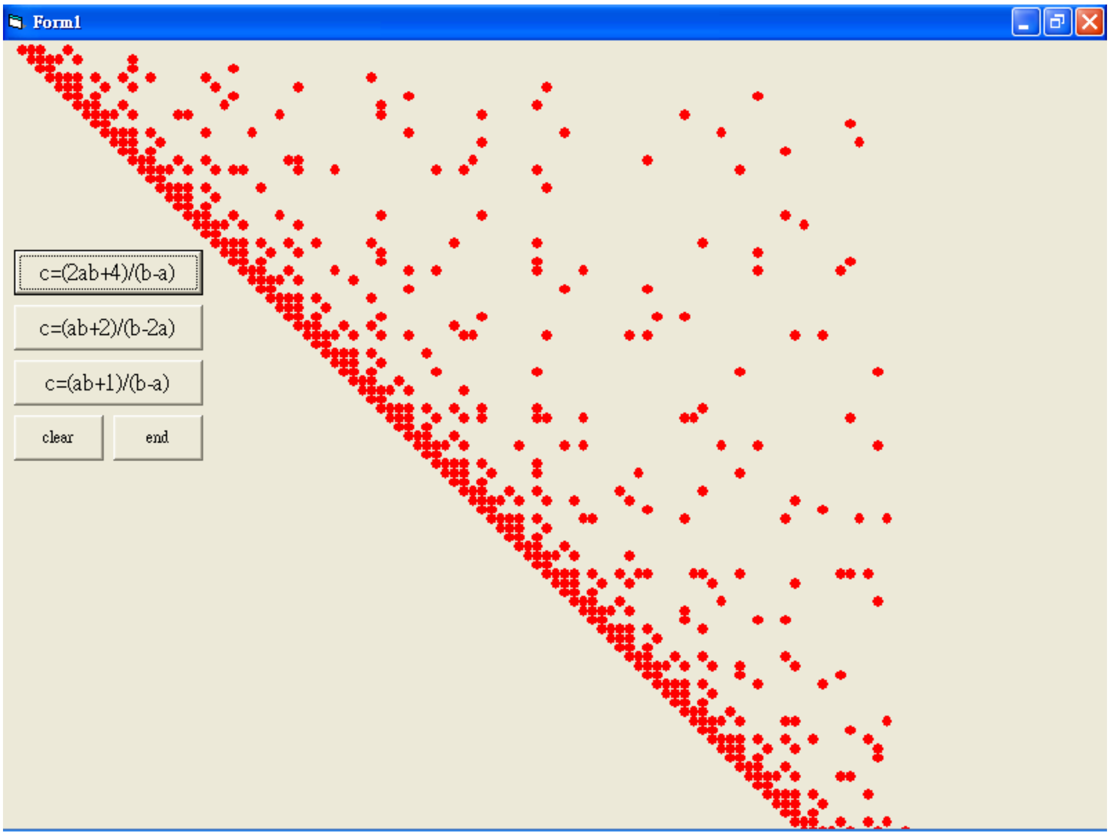



***Fig 11.*** ***Computer solutions of*** $ab + 2ac + 2 = bc$

| a | b | c |
|---|---|---|
| $F_{2n}$ | $F_{2n+1}$ | $F_{2n+2}$ |
| $n$ | $n+1$ | $n^2+n+1$ |
| $2n+1$ | $2n+3$ | $2n^2+4n+2$ |
| $F_{2n+2}/2$ | $F_{2n+1}$ | $F_{2n+4}$ |
| $n$ | $n+1$ | $2n^2+2n+4$ |
| $n$ | $n+2$ | $n^2+2n+2$ |
| $F_{2n}$ | $F_{2n+1}/2$ | $F_{2n+3}$ |
| $n$ | $2n+1$ | $2n^2+n+2$ |
| $n$ | $2n+2$ | $n^2+n+1$ |

***Table 1 Patterns of solutions***

## III. Conclusion

The research covers several mathematical fields, including geometry, algebra, trigonometry and number theory. In the first part, analytic geometry and trigonometric algebra are applied to prove Hayashi's three identities. We also succeed in generalizing equations of Fibonacci numbers to equations of Lucas numbers and general second-order liner recursive sequences. So far as we know, nothing like this has been done. Now, by means of similar methods, we aim to obtain positive integer solutions to $ab+ac+1=bc$, $ac+2ab+4=bc$ and $ab+2ac+2=bc$.

Let a, b, c, be positive integers. If $ab+ac+1=bc$, $ac+2ab+4=bc$ and $ab+2ac+2=bc$, then $\cot^{-1}b+\cot^{-1}c=\cot^{-1}a$, $\cot^{-1}b+\cot^{-1}c=\cot^{-1}(a/2)$, and $\cot^{-1}(b/2)+\cot^{-1}c=\cot^{-1}a$. What are the solutions that can satisfy $ab+ac+1=bc$, $ac+2ab+4=bc$, and $ab+2ac+2=bc$? We obtain the regulation table as follows:

| a | b | c |
|---|---|---|
| $F_{2n}$ | $F_{2n+1}$ | $F_{2n+2}$ |
| $n$ | $n+1$ | $n^2+n+1$ |
| $2n+1$ | $2n+3$ | $2n^2+4n+2$ |
| $F_{2n+2}/2$ | $F_{2n+1}$ | $F_{2n+4}$ |
| $n$ | $n+1$ | $2n^2+2n+4$ |
| $n$ | $n+2$ | $n^2+2n+2$ |
| $F_{2n}$ | $F_{2n+1}/2$ | $F_{2n+3}$ |
| $n$ | $2n+1$ | $2n^2+n+2$ |
| $n$ | $2n+2$ | $n^2+n+1$ |

## IV. Future Investigation

Still, after this research is done, we find several potential topics for further study.

Visualization of solutions to Diophantine equations appears promising. The following remaining works are to be studied in future:

1. The classification of all the integer solutions satisfying the following equations:

   (1) $\cot^{-1} b + \cot^{-1} c = \cot^{-1} a$ ;
   (2) $\cot^{-1} b + \cot^{-1} c = \cot^{-1}(a/2)$ ;
   (3) $\cot^{-1}(b/2) + \cot^{-1} c = \cot^{-1} a$ ;
   (4) $\cot^{-1} b + \cot^{-1} c = \cot^{-1}(p/q)$ ;
   (5) $\cot^{-1} a + \cot^{-1}(b/c) = \cot^{-1}(d/e) + \cot^{-1}(f/g)$ .

2. 3-D visualization of the solutions of the equation $\cot^{-1} y + \cot^{-1} z = \cot^{-1} x$ .

3. This research combines the power of several branches of mathematics. The stimulation derived from our work exceeds the sum from individual subject matters taken together. We trust that many more topics in mathematics can be explored this way.

## V. References


[1] Ko Hayashi, Fibonacci Numbers and the Arc-cotangent Function, Math. Magazine, Vol.76, (3), pp.214-215, July 2003.

[2]http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibFormulae.html#order2luc

[3] L. S. Johnston, The Fibonacci Sequence and Allied Trigonometric Identities, Amer. Math. Monthly, Vol. 47, (2), pp. 85-89, Feb., 1940.

Accessories 1:

```
graphics = 88, 116, 520, 462, , 66, 87, 498, 433, C
pset = 110, 145, 537, 491, Z, 88, 116, 515, 462, C
```

```
VERSION 5.00
Begin VB.Form Form1
   AutoRedraw      =   -1  'True
   Caption         =   "Form1"
   ClientHeight    =   4545
   ClientLeft      =   60
   ClientTop       =   450
   ClientWidth     =   6630
   LinkTopic       =   "Form1"
   ScaleHeight     =   5
   ScaleMode       =   0  '使用者自訂
   ScaleWidth      =   5
End
Attribute VB_Name = "Form1"
Attribute VB_GlobalNameSpace = False
Attribute VB_Creatable = False
Attribute VB_PredeclaredId = True
Attribute VB_Exposed = False
Dim a, b As Integer

Dim c, j As Integer

Dim arr(100, 100) As String
Private Sub Form_Activate()

     For a = 1 To 100
        For b = 1 To 100
           If (b - a) > 0 Then
              c = (a * b + 1) / (b - a)

              If Int(c) = c Then
                 arr(b, a) = "O"
```

```
            Else
               arr(b, a) = "X"
            End If

         Else
            arr(b, a) = "X"
         End If

      Next
   Next

   Print Space(2); Str(1);
   For i = 2 To 100
       Print Space(1); Str(i);
   Next

   Print

   For b = 1 To 100
       Print Space(Int(b / 10)); Str(b);

       For a = 1 To 100
         Print Space(2); arr(b, a);
          'If arr(b, a) = "X" Then
          '   PSet (b, a), RGB(255, 0, 0)
          'End If
       Next
       Print
   Next

Form2.Show

End Sub
```

```
VERSION 5.00
Begin VB.Form graphics
   Caption         =   "graphics"
```

```
    ClientHeight     =    6420
    ClientLeft       =    60
    ClientTop        =    450
    ClientWidth      =    8115
    LinkTopic        =    "Form2"
    ScaleHeight      =    2
    ScaleMode        =    0   '使用者自訂
    ScaleWidth       =    2
    StartUpPosition =    3   '系統預設值
    Begin VB.CommandButton Command1
       Caption          =    "關閉"
       Height           =    615
       Left             =    4200
       TabIndex         =    0
       Top              =    5640
       Width            =    1335
    End
End
Attribute VB_Name = "graphics"
Attribute VB_GlobalNameSpace = False
Attribute VB_Creatable = False
Attribute VB_PredeclaredId = True
Attribute VB_Exposed = False
Dim a, b As Integer

Dim c, j As Integer

Dim arr(100, 100) As Integer
Private Sub Form_Activate()

     For a = 1 To 100
        For b = 1 To 100
           If (b - a) > 0 Then
              c = (a * b + 1) / (b - a)

              If Int(c) = c Then
                 arr(b, a) = 1
              Else
```

```
                arr(b, a) = 0
            End If

        Else
            arr(b, a) = 0
        End If

    Next
Next



End Sub

Private Sub Form_Paint()
    ScaleMode = 6

     For b = 1 To 100
        For a = 1 To 100
            If arr(b, a) = 1 Then
                PSet (3 * b, 3 * a), RGB(255, 0, 0)
            End If
        Next
        Print
    Next
End Sub

Private Sub Command1_Click()
    Me.Hide
End Sub
```

# 評語及建議事項

1. 優點：結合幾何、代數、三角、數論、遞迴於同一件作品，是難得的好題材！
2. 優點：版面精緻。
3. 優點：報告時顯示出清楚的概念。